Math 4803 February 28, 2024

LATELY

Lots of examples of <u>surfaces</u>,

and a few <u>pillowcases</u>.

TODAY

<u>Tessellations</u> of $(\mathbb{R}^2, d_{euc})$, $(\mathbb{H}^2, d_{hyp})$,

and (S^2, d_{sph}).

Tessellations

Let X be the Euclidean plane, the hyperbolic plane, or the sphere.

A tessellation of X is a family of tiles X_n, $n \in \mathbb{N}$ such that

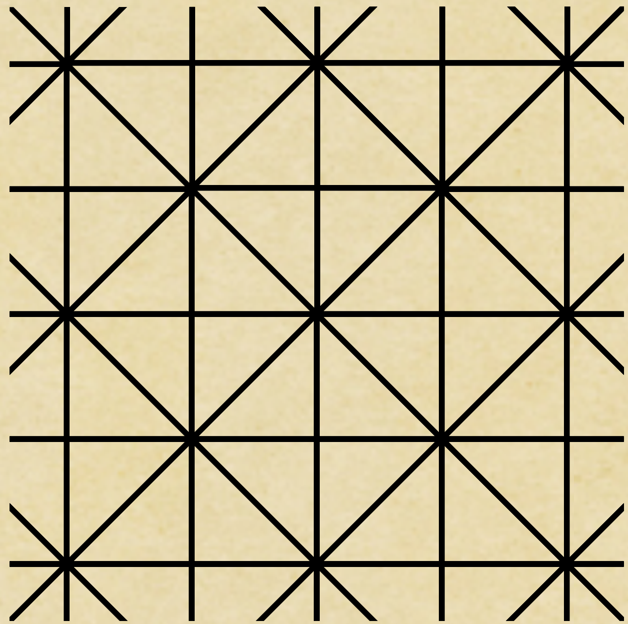
(1) each tile is a connected polygon in X;

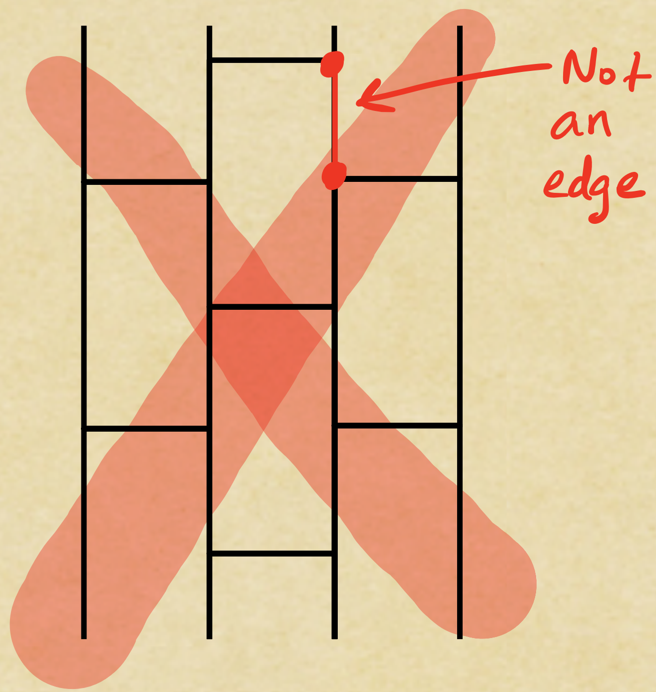
(2) the tiles are pairwise isometric;

(3) the union of the tiles is X;

(4) for $m \neq n$, $X_m \cap X_n$ consists only of edges & vertices of X_m, and these are shared with X_n.

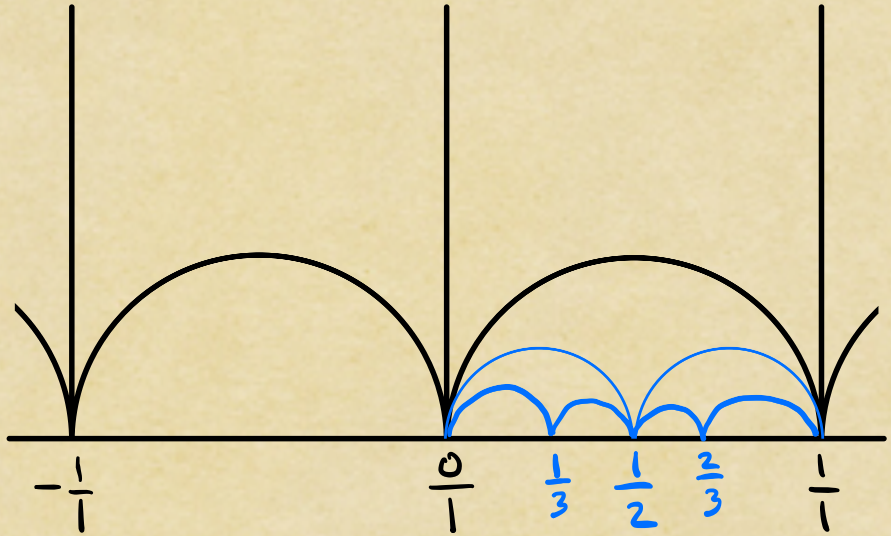
(5) for every $P \in X$, there exists $\varepsilon > 0$ s.t. $\{n \in \mathbb{N} \mid B_d(P, \varepsilon) \cap X_n \neq \emptyset\}$ is finite. (local finiteness)



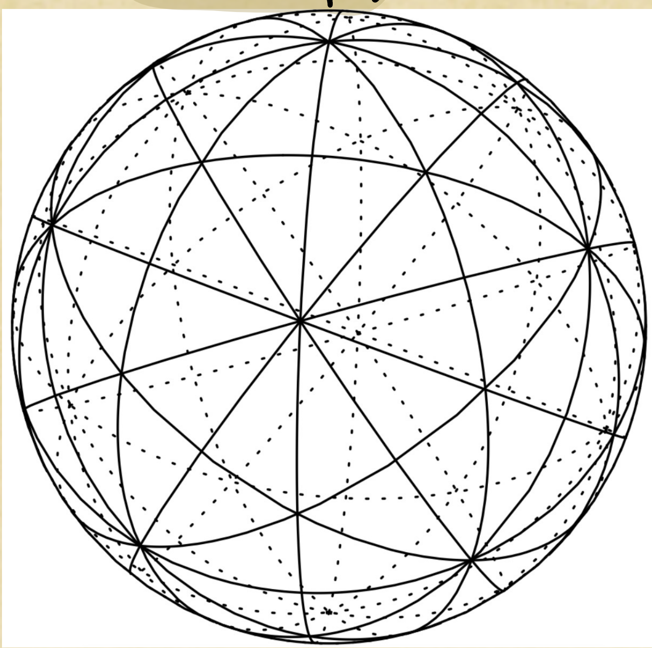


Examples

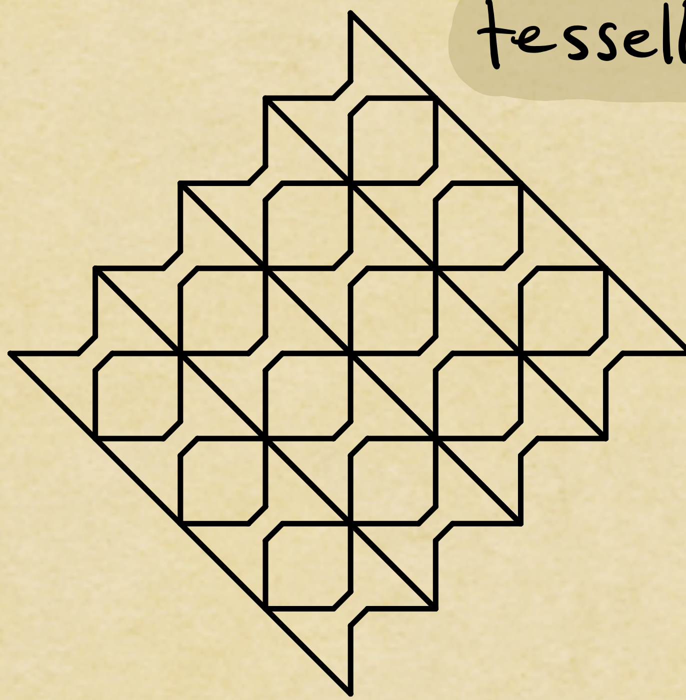
Farey tessellation of $(\mathbb{H}^2, d_{hyp})$



A tessellation of (S^2, d_{sph})

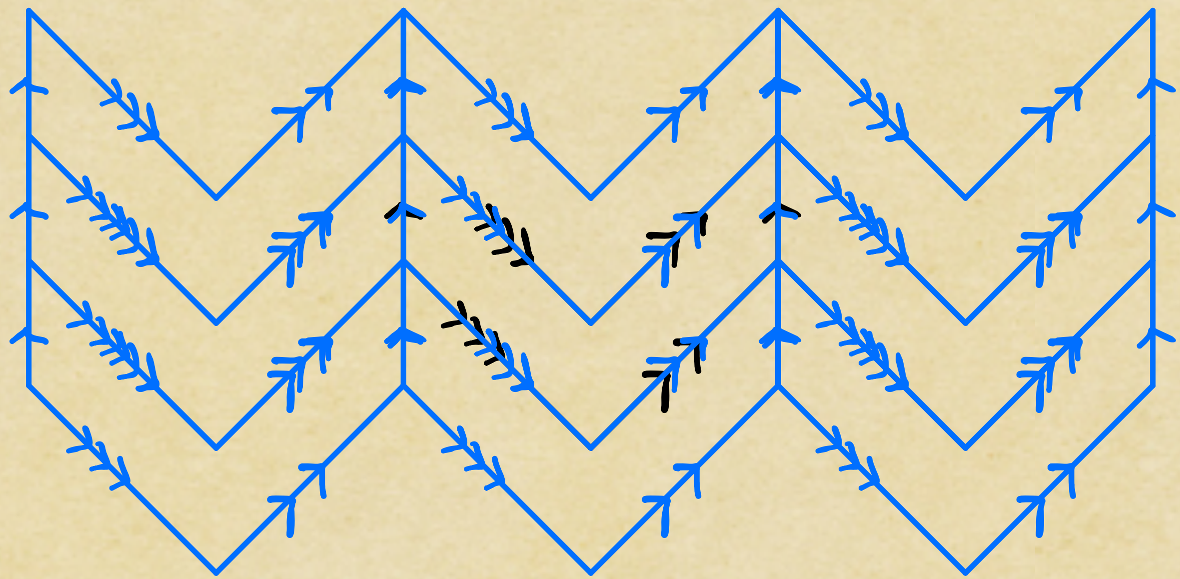


Not technically a tessellation



The tiling group

We will build tessellations from polygons with edge gluings.



Consider a polygon X in the Euclidean plane, hyperbolic plane, or sphere, and suppose we have an edge gluing $\{\varphi_i : E_i \to E_{i\pm1} \mid 1 \le i \le 2k\}$. Each φ_i extends to a(n) unique isometry of the full space such that $\varphi_i(X)$ & X are on opposite sides of $\varphi_i(E_i)$. The tiling group of this edge gluing is then the subgroup of the full isometry group which is generated by these extensions, denoted Γ.

The tessellation theorem

Thm. Let X be a connected polygon in the Euclidean plane, hyperbolic plane, or sphere, and suppose that an edge gluing $\{\varphi_i : E_i \to E_{i\pm 1}\}$ has been specified. If

(1) for every vertex $P \in X$, $\sum_{Q \sim P} \measuredangle(Q) = \frac{2\pi}{n}$, where $n > 0$ is an integer which may depend on P;

(2) the quotient metric space $(\overline{X}, \overline{d}_X)$ is complete;

then the family $\{\varphi(X) \mid \varphi \in \Gamma\}$ is a tessellation of the Euclidean plane, hyperbolic plane, or sphere.

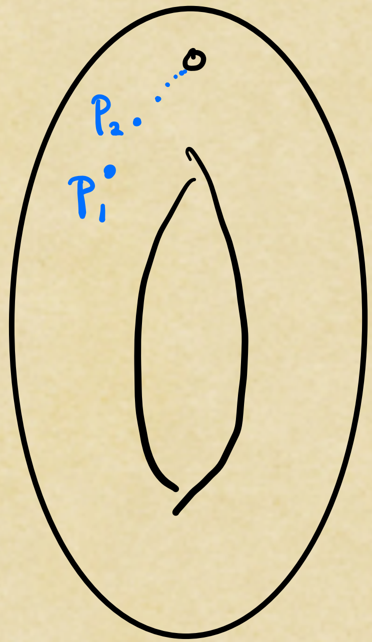
Completeness

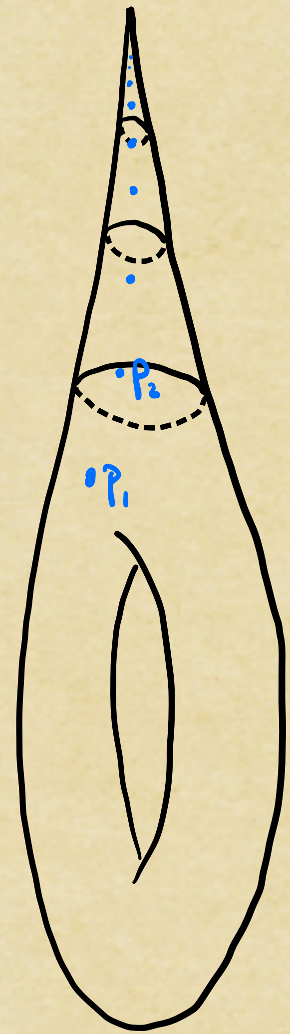
Let (X,d) be a metric space.

The length of a sequence $P_1, P_2, \dots, P_n, \dots$ is defined to be

$$\sum_{n=1}^{\infty} d(P_n, P_{n+1})$$

if this series converges, and ∞ otherwise.





We say that (X,d) is complete if every sequence of points in X which has finite length converges to a point $P_\infty \in X$.

HW: A metric space (X,d) is complete if and only if all of its Cauchy sequences converge.

After the midterm

- Proof of the tessellation theorem
- Analysis results (Completeness + compactness)
- Examples!