Math 4803 February 26, 2024

LAST TIME

The pseudosphere

TODAY

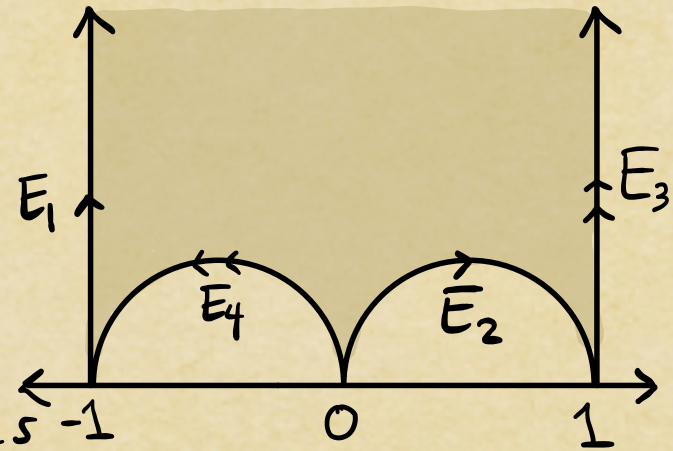
The once-punctured torus

& triangular pillowcases

The once-punctured torus

We'll now construct a hyperbolic surface $(\overline{X}, \overline{d}_X)$ with the property that, for all $\overline{P}, \overline{Q} \in \overline{X}$, $\exists\ P \overset{\gamma}{\rightsquigarrow} Q$ s.t.

$$\ell_{\overline{d}_X}(\gamma) = \overline{d}_X(P, Q)$$



The polygon X is bounded by $E_1 = \{\operatorname{Re} z = -1\}$, $E_2 = \{|z - \frac{1}{2}| = \frac{1}{2}\}$, $E_3 = \{\operatorname{Re} z = 1\}$, & $E_4 = \{|z + \frac{1}{2}| = \frac{1}{2}\}$.

This is an "ideal square" — its vertices aren't actually in $\mathbb{H}^2$. We define an edge gluing by

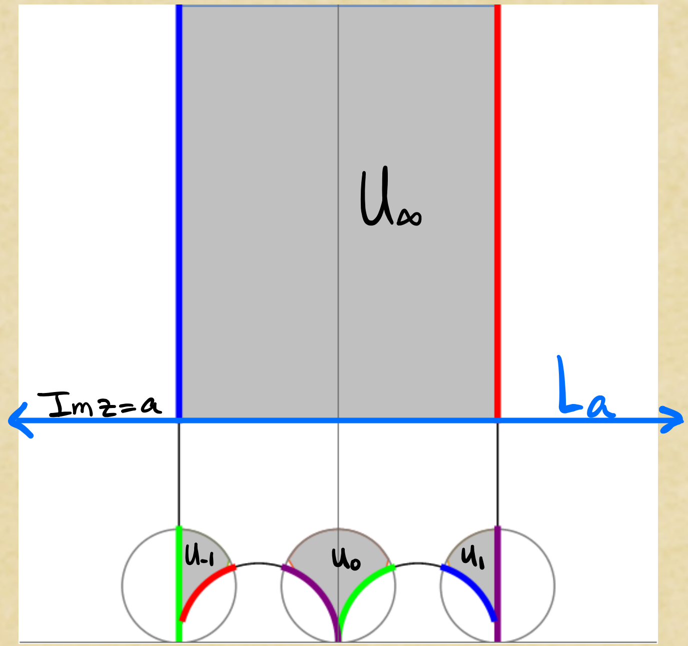
$$\varphi_1 : E_1 \longrightarrow E_2 \qquad \& \qquad \varphi_3 : E_3 \longrightarrow E_4$$

$$z \longmapsto \frac{z+1}{z+2} \qquad\qquad z \longmapsto \frac{z-1}{-z+2}$$

The quotient $(\overline{X}, \overline{d}_X)$ is homeomorphic to $T^2 - \{*\}$.

The once-punctured torus

Notice that, despite the puncture, $(\overline{X}, \overline{d}_X)$ is <u>geodesically complete</u> — for any $\overline{P}, \overline{Q} \in \overline{X}$, we can find $\overline{P} \overset{\gamma}{\rightsquigarrow} \overline{Q}$ s.t. $\overline{\ell_{hyp}}(\gamma) = \overline{d}_X(\overline{P}, \overline{Q})$. This indicates some strange behavior near the puncture, so let's investigate.



Given $a > 1$ we can consider

$$U_\infty = \{ z \in X \mid \operatorname{Im} z \geq a \}.$$

This will give one <u>sectors</u> of a "disc" around the puncture. To determine the other sectors, consider the line

$$L_a = \{ z \in \mathbb{H}^2 \mid \operatorname{Im} z = a \}.$$

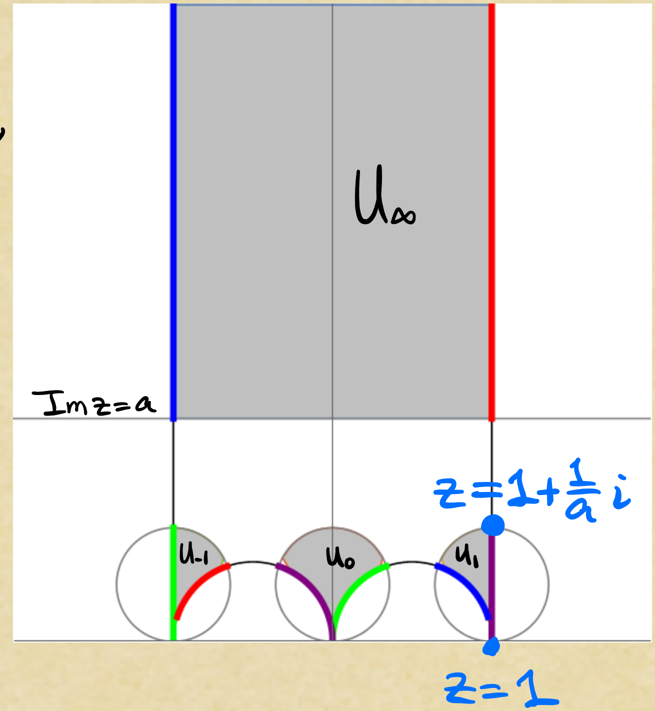
The other sectors will lie on the "puncture side" of $\varphi_1(L_a)$, $\varphi_3(L_a)$, or $(\varphi_1 \circ \varphi_3)(L_a) = (\varphi_3 \circ \varphi_1)(L_a)$.

The once-punctured torus

Group work Check that $\varphi_1(L_a)$, $\varphi_3(L_a)$, and $(\varphi_1 \circ \varphi_3)(L_a) = (\varphi_3 \circ \varphi_1)(L_a)$ are the (Euclidean) circles of radius $1/2a$

centered at
- $1 + \frac{1}{2a} i$;
- $-1 + \frac{1}{2a} i$;
- $\frac{1}{2a} i$;

respectively.



Recall: LFMs take circles & lines to circles & lines.

So $\varphi_1(L_a)$ will be a circle tangent to $\varphi_1(\infty) = 1$.

Now $\varphi_1^{-1}(z) = \dfrac{2z-1}{-z+1}$.

The once-punctured torus

Group work, cont'd

So $\varphi_1^{-1}\left(1+\frac{1}{a}i\right) = \dfrac{2\left(1+\frac{1}{a}i\right)-1}{-\left(1+\frac{1}{a}i\right)+1}$

$$= \frac{2a+2i-a}{-a-i+a} = \frac{a+2i}{-i}$$

$$= -2+ai$$

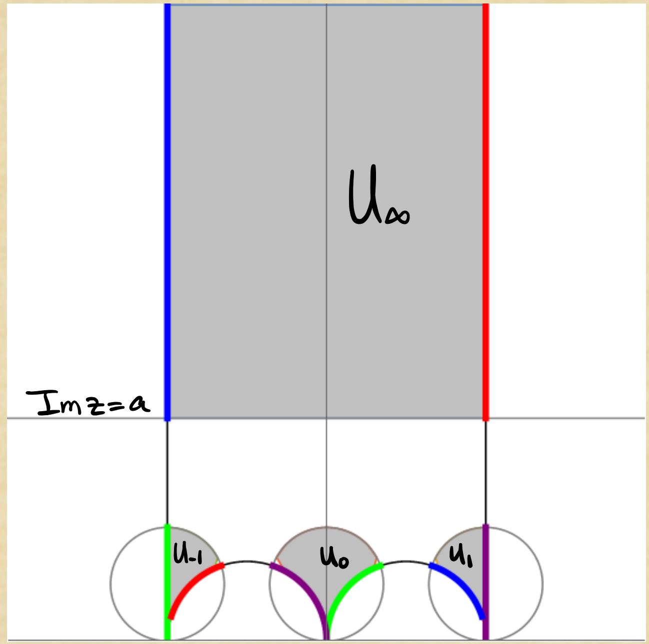
So $\operatorname{Im}\left(\varphi_1^{-1}\left(1+\frac{1}{a}i\right)\right) = a$.

Namely, $\varphi_1^{-1}\left(1+\frac{1}{a}i\right) \in L_a$.

$\therefore$ $\varphi_1(L_a)$ intersects the desired circle, and thus *is* the desired circle.

The once-punctured torus

We can now define U_∞, U_{-1}, U_0, & U_1 as in the figure, let $U = U_\infty \cup U_{-1} \cup U_0 \cup U_1$, and consider the image $\overline{U}$ in $\overline{X}$.



(We needed to think about $\varphi_1(L_a)$, $\varphi_3(L_a)$, $(\varphi_1 \circ \varphi_3)(L_a)$, and $(\varphi_3 \circ \varphi_1)(L_a)$ in order for these to glue nicely.)

This quotient $\overline{U}$ is our "disk around the puncture", and it turns out to be isometric to an end of the pseudosphere

Given $a > 1$, we let S_a denote the surface obtained by revolving the curve $t \mapsto (t - \tanh t, \operatorname{sech} t)$,

$$\cosh^{-1}\left(\frac{a\pi}{3}\right) \le t < \infty$$

about the x-axis. Notice that $S_a \subset S$.

The once-punctured torus

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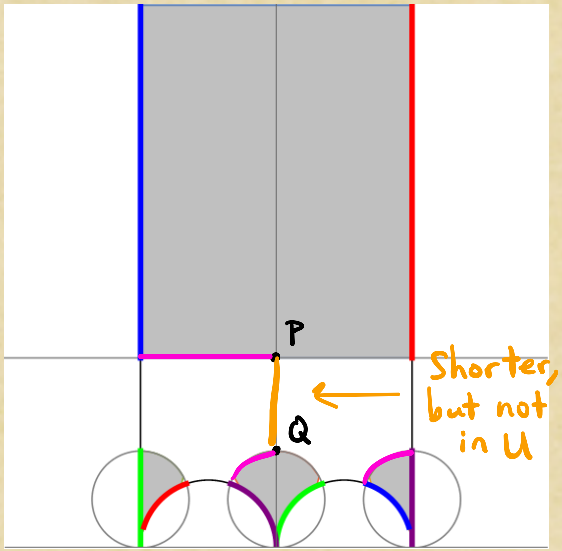
We let d_{S_a} denote the Euclidean path metric on S_a.

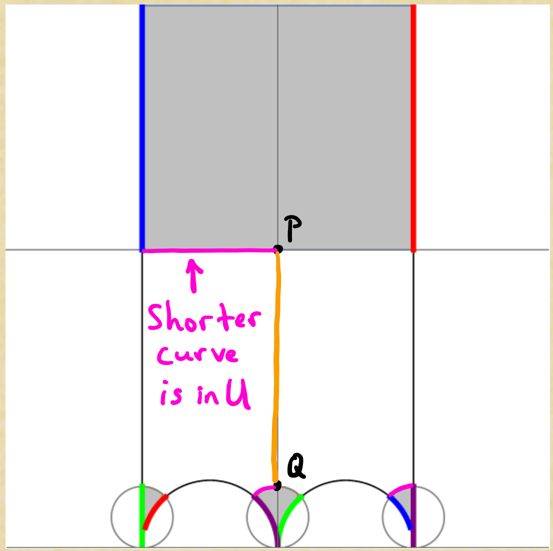
Prop. The metric spaces $(\overline{U}, \overline{d}_U)$ & (S_a, d_{S_a}) are isometric.

(Proof idea.) It's the same basic idea as in the hyperbolic cylinder case: write down an explicit map $U \to S_a$ which preserves arclengths, then verify that it plays nicely with the quotient. Messier because there are now more pieces. $\diamond$

The once-punctured torus

Finally, while we have $(\overline{U}, \overline{d_U}) \simeq (S_a, d_{S_a})$ for any $a>1$, $(\overline{U}, \overline{d_U})$ is not a subspace of $(\overline{X}, \overline{d_X})$ if a is too close to 1.





i.e., if $a \approx 1$, then $\overline{d_X}(\overline{P}, \overline{Q}) < \overline{d_U}(\overline{P}, \overline{Q})$.

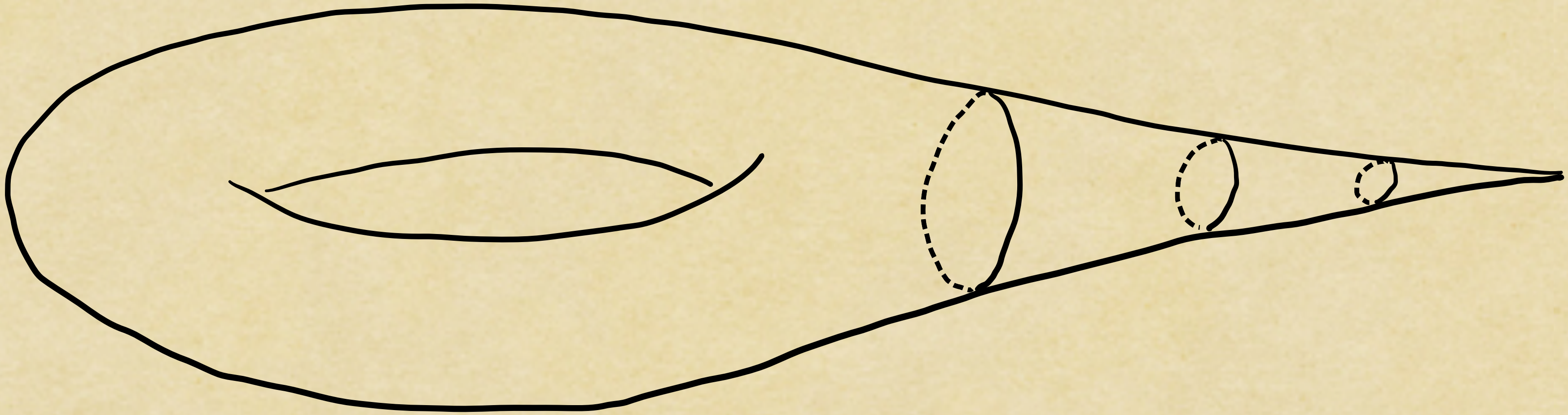
(We always have $\overline{d_X}(\overline{P}, \overline{Q}) \leq \overline{d_U}(\overline{P}, \overline{Q})$.)

We can fix this by choosing a sufficiently large.

<u>Prop.</u> If $a \cdot \ln(a) > \frac{3}{2}$, then $\overline{d_U} = \overline{d_X}|_{\overline{U}}$.

Informal corollary. $T^2 - \{*\}$ looks like a pseudosphere near its puncture.

The once-punctured torus



Triangular pillowcases

Our last examples will have cone singularities, and thus fail to be everywhere locally isometric to one of $(\mathbb{R}^2, d_{euc})$, $(\mathbb{H}^2, d_{hyp})$, & (S^2, d_{sph}).

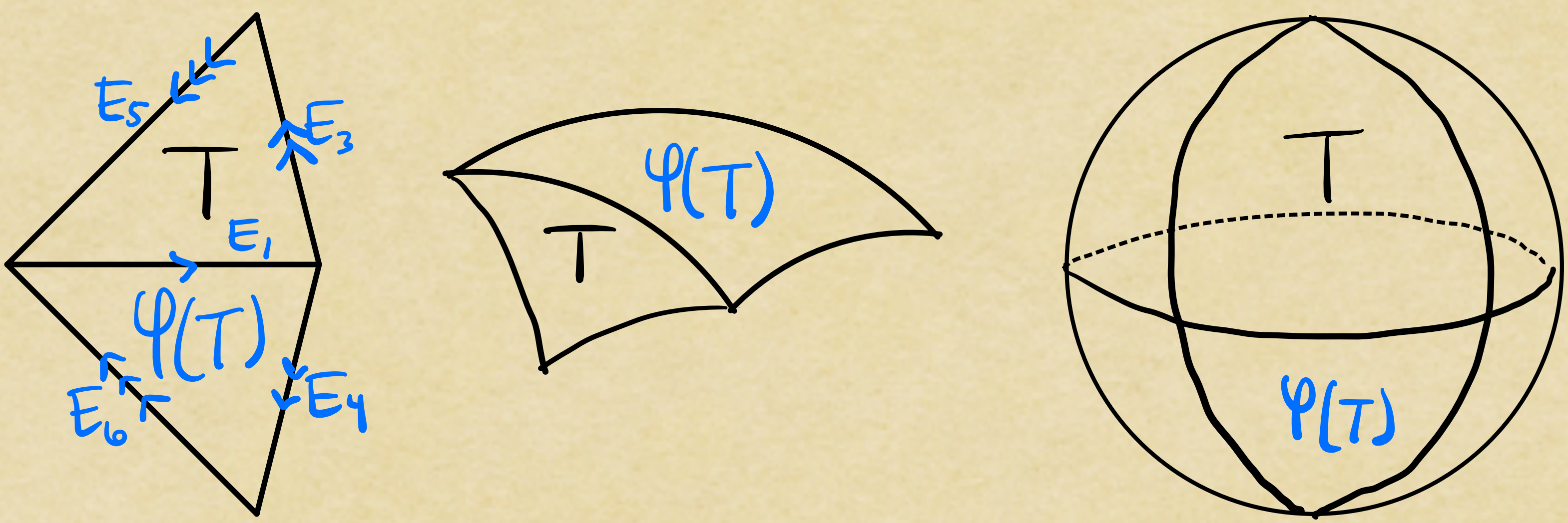
We first need:

Prop Consider $\alpha, \beta, \gamma \in (0, \pi)$.

1. If $\alpha+\beta+\gamma=\pi$, then there is a triangle T in $(\mathbb{R}^2, d_{euc})$ with area 1 and angles α, β, γ.
2. If $\alpha+\beta+\gamma<\pi$, then there is a triangle T in $(\mathbb{H}^2, d_{hyp})$ with angles α, β, γ.
3. If $\pi<\alpha+\beta+\gamma<\pi+2\min\{\alpha,\beta,\gamma\}$, then there is a triangle T in (S^2, d_{sph}) with angles α, β, γ.

Triangular pillowcases

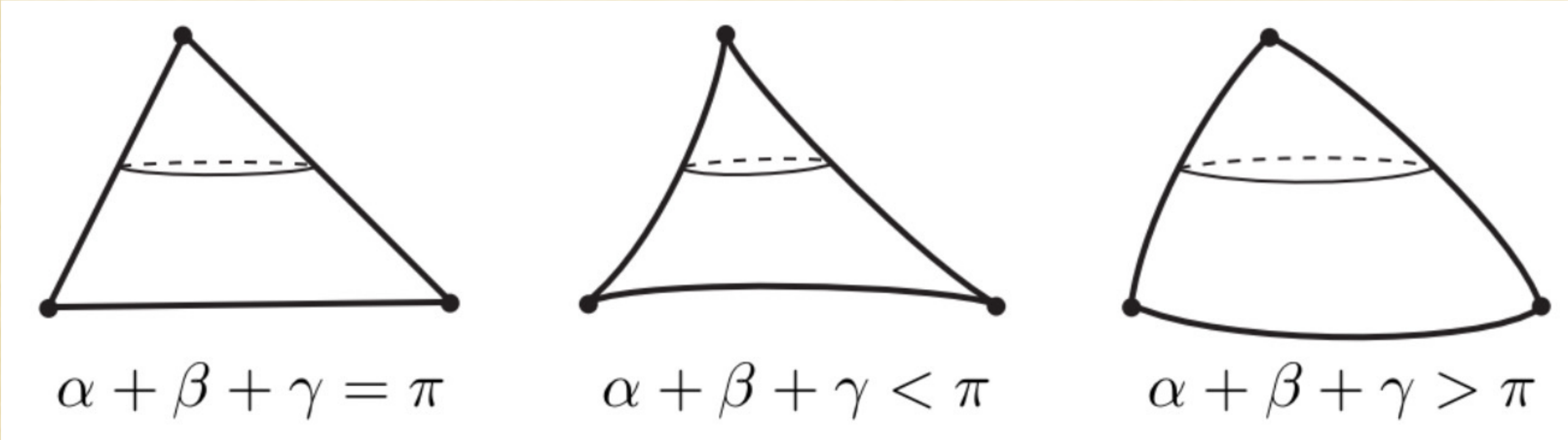
Given $\alpha, \beta, \gamma \in (0, \pi)$ s.t. $\alpha+\beta+\gamma < \pi + 2\min\{\alpha,\beta,\gamma\}$, let T be the triangle provided by the proposition, and let φ be the relevant version of "reflection over a side."



We then let $X = T \cup \varphi(T)$, treating T & $\varphi(T)$ as disjoint polygons. We can label the edges of T E_1, E_3, E_5 and set $E_2 = \varphi(E_1)$, $E_4 = \varphi(E_3)$, $E_6 = \varphi(E_5)$. Then φ automatically gives us an edge gluing.

Triangular pillowcases

The resulting quotient $(\overline{X}, \overline{d_X})$ is necessarily a metric space, but is not a Euclidean, hyperbolic, or spherical surface.



Instead, $(\overline{X}, \overline{d}_X)$ is locally isometric to one of $(\mathbb{R}^2, d_{eucl})$, $(\mathbb{H}^2, d_{hyp})$, or (S^2, d_{sph}) at all but three points, where there are cone angles of $2\alpha, 2\beta, \& \, 2\gamma$ $< 2\pi$.