Math 4803 February 21, 2024

LAST TIME

Euclidean & hyperbolic

Cylinders & Möbius strips

TODAY

The pseudosphere & the

once-punctured torus.

The pseudosphere

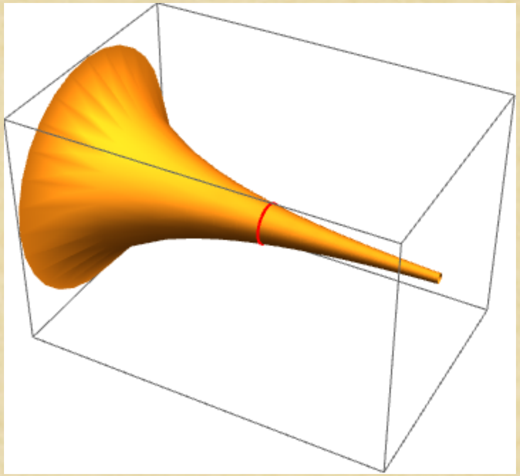
The tractrix is the curve parametrized by

$t \mapsto (t - \tanh t, \operatorname{sech} t)$, $0 \le t < \infty$.

We obtain the pseudosphere by revolving this about the x-axis in $\mathbb{R}^3$: for $0 \le s < 2\pi$, $0 \le t < \infty$,

$$(s,t) \mapsto (t - \tanh t, \operatorname{sech} t \cdot \cos s, \operatorname{sech} t \cdot \sin s).$$

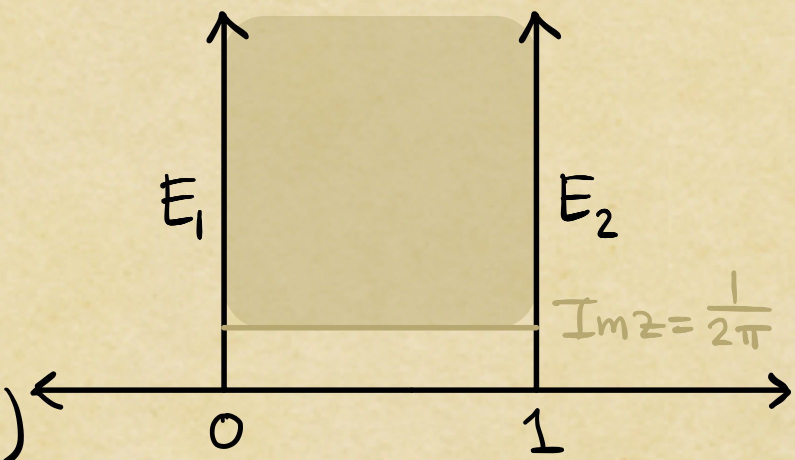
On this surface S we let d_S denote the Euclidean path metric.



Aside: We call (S, d_S) the pseudosphere b/c it has constant Gaussian curvature -1.

The pseudosphere

Let
$$X^+ = \left\{ z \;\middle|\; \begin{matrix} 0 \le \operatorname{Re} z \le 1 \\ \operatorname{Im} z \ge \frac{1}{2\pi} \end{matrix} \right\} \subset \mathbb{H}^2,$$



and let $d^+ = d_{hyp}|_{X^+}$. Let $(\overline{X^+}, \overline{d^+})$ be the hyperbolic surface obtained via $\varphi_1 : E_1 \to E_2$, $z \mapsto z+1$.

<u>Prop.</u> The quotient $(\overline{X^+}, \overline{d^+})$ is <u>isometric to</u> (S, d_S).

(Proof idea.) We can parametrize S by
$$\begin{aligned}[0,1]\times[0,\infty) &\longrightarrow S \\ (s,t) &\longmapsto (t - \tanh t,\ \operatorname{sech} t \cdot \cos(2\pi s),\ \operatorname{sech} t \cdot \sin(2\pi s))\end{aligned}$$

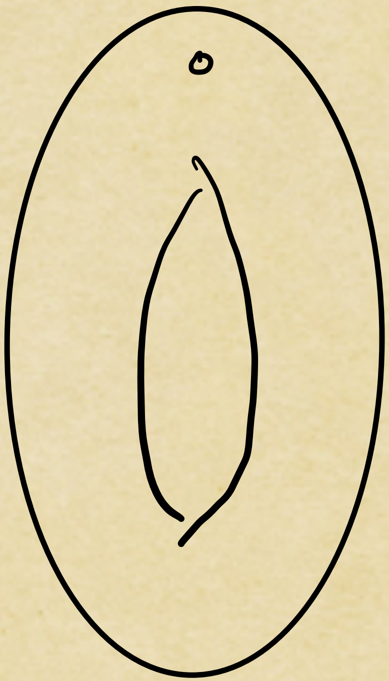
Precomposing with
$$\begin{aligned} X^+ &\longrightarrow [0,1]\times[0,\infty) \\ z &\longmapsto (\operatorname{Re} z, \operatorname{arccosh}(2\pi \operatorname{Im}(z)))\end{aligned}$$
gives us

$\overline{\rho} : \overline{X^+} \longrightarrow S$, a bijection s.t. $\ell_{euc}(\overline{\rho}(\gamma)) = \ell_{hyp}(\gamma)$, for every p.w.d. curve γ in X^+.

$\diamond$

The once-punctured torus

The **once-punctured torus** is defined to be $T^2 - \{*\}$, where T^2 is the torus and $* \in T^2$ is an arbitrary point. This definition only determines the homeomorphism type of $T^2 - \{*\}$. We also want a metric space structure.



Easy option: $d_{T^2 - \{*\}} = \overline{d}_{euc} \big|_{T^2 - \{*\}}$

i.e., $d_{T^2 - \{*\}}(P, Q) := d_{T^2}(P, Q)$

Problem: We prefer metrics which are "geodesically complete."

The once-punctured torus

We'll now construct a hyperbolic surface $(\overline{X}, \overline{d}_X)$ with the property that, for all $\overline{P}, \overline{Q} \in \overline{X}$, $\exists$ $P \overset{\gamma}{\rightsquigarrow} Q$ s.t.

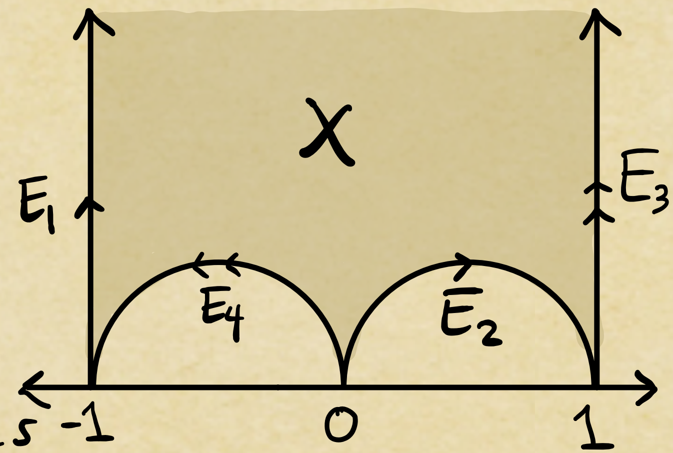
homeo. to $T^2 - \{*\}$

$$\ell_{\overline{d}}(\gamma) = \overline{d}(\overline{P}, \overline{Q}).$$

The polygon X is bounded by

$E_1 = \{\mathrm{Re}\, z = -1\}$, $E_2 = \{|z - \tfrac{1}{2}| = \tfrac{1}{2}\}$,

$E_3 = \{\mathrm{Re}\, z = 1\}$, & $E_4 = \{|z + \tfrac{1}{2}| = \tfrac{1}{2}\}$.



This is an "ideal square" — its vertices aren't actually in $\mathbb{H}^2$. We define an edge gluing by

$$\varphi_1 : E_1 \longrightarrow E_2 \qquad \qquad \varphi_3 : E_3 \longrightarrow E_4$$

$$z \longmapsto \frac{z+1}{z+2} \qquad \qquad z \longmapsto \frac{z-1}{-z+2}$$

The quotient $(\overline{X}, \overline{d}_X)$ is homeomorphic to $T^2 - \{*\}$.