Math 4803 February 19, 2024

LAST TIME

Euclidean surfaces from hexagons & the surface of genus two as a quotient a regular octagon, up to homeomorphism.

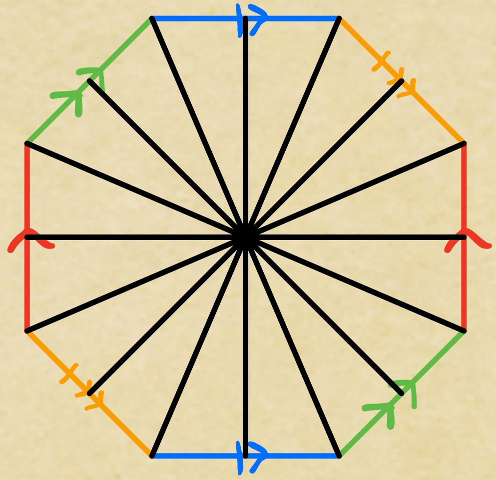
TODAY

The surface of genus two as a hyperbolic surface, the projective plane, & some non-compact surfaces.

The surface of genus two

Later we'll show that there is no Euclidean surface homeomorphic to the surface of genus two (Σ_2).

But we can explicitly construct a hyperbolic surface homeomorphic to Σ_2.



Idea: We want each triangle seen on the left to have angles $\frac{\pi}{2}, \frac{\pi}{8}, \; \frac{\pi}{8}$.

Certainly we can't accomplish this in Euclidean or Spherical geometry, but we'll build a hyperbolic triangle of this type.

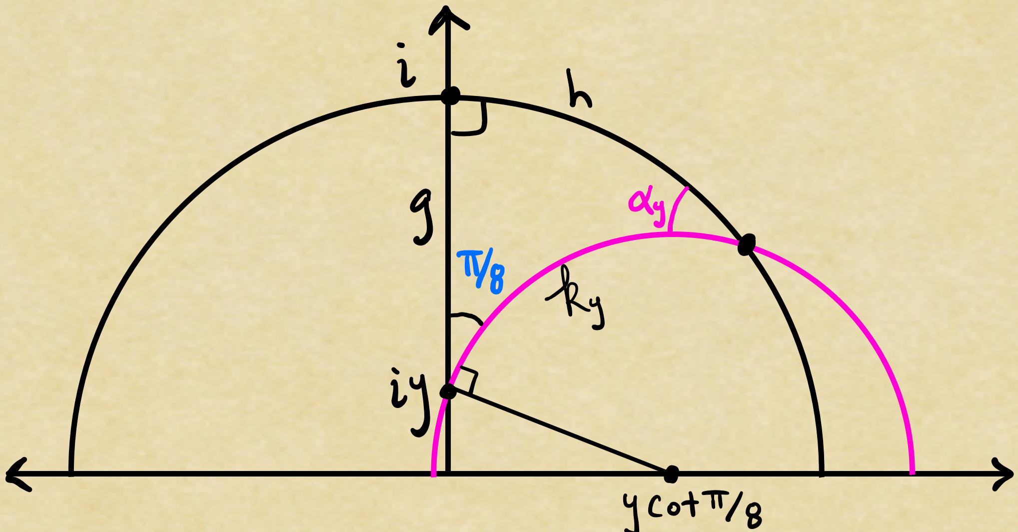
The surface of genus two

<u>Lemma</u>. In $(\mathbb{H}^2, d_{hyp})$ there is a triangle with angles $\frac{\pi}{2}$, $\frac{\pi}{8}$, & $\frac{\pi}{8}$.

(Proof.) We'll let g & h be the complete geodesics

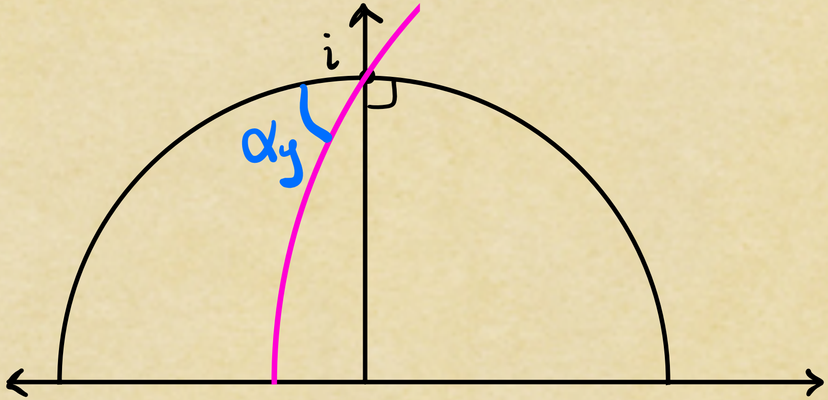
$$g = \{\mathrm{Re}\, z = 0\} \quad \& \quad h = \{|z| = 1\}$$

in $\mathbb{H}^2$.

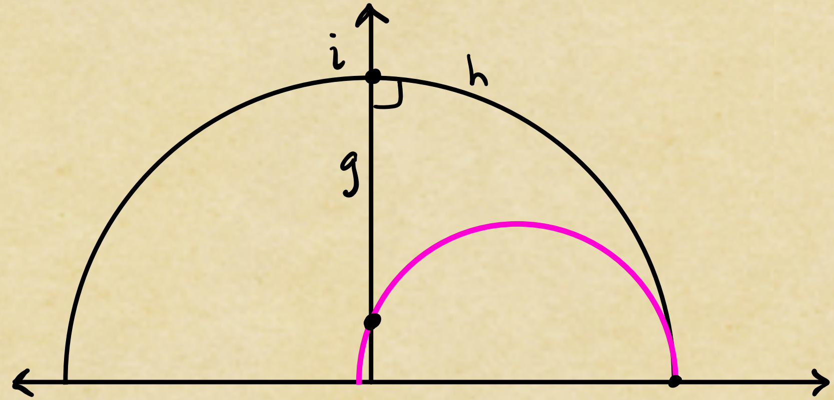


For $0 < y \leq 1$ we'll let k_y be the complete geodesic which makes an angle of $\pi/8$ with g at iy and is centered on the positive x-axis.

The surface of genus two



$y = 1$



$y = (\sin \pi/8)/(1 + \cos \pi/8)$

If $y > \overbrace{(\sin \frac{\pi}{8})/(1 + \cos \frac{\pi}{8})}^{y_{min}}$, then k_y will intersect h, and we let α_y denote the angle of intersection.

When $y = 1$, $\alpha_y = \frac{\pi}{2} - \frac{\pi}{8} = \frac{3\pi}{8}$ and

$$\lim_{y \to y_{min}^{+}} \alpha_y = 0.$$

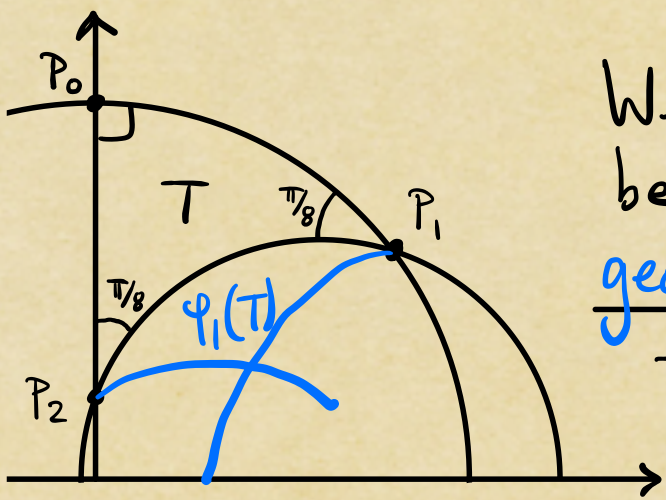
By the IVT, there must be some $y < 1$ s.t. $\alpha_y = \frac{\pi}{8}$. ◇

The surface of genus two

We're now ready to put a hyperbolic metric on Σ_2.

Proposition. In $(\mathbb{H}^2, d_{hyp})$ there is an octagon whose sides all have the same length and whose angles are all $\frac{\pi}{4}$.

(Proof.) Throughout, T is the triangle constructed above.

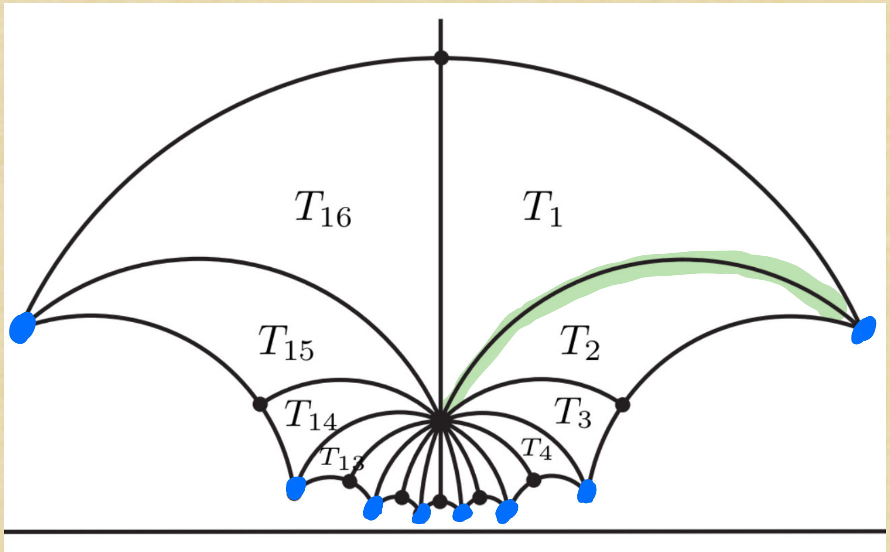


We let $T_1 = T$ and let $\varphi_1 : \mathbb{H}^2 \to \mathbb{H}^2$ be inversion across the complete geodesic thru P_1 & P_2. We then let $T_2 = \varphi_1(T_1)$ and define $\varphi_2 : \mathbb{H}^2 \to \mathbb{H}^2$ to be inversion across the complete geodesic thru $\varphi_1(P_0)$ & P_2.

We proceed inductively, alternating the side over which we invert, until we've defined $T_1, T_2, \ldots, T_{16}$.

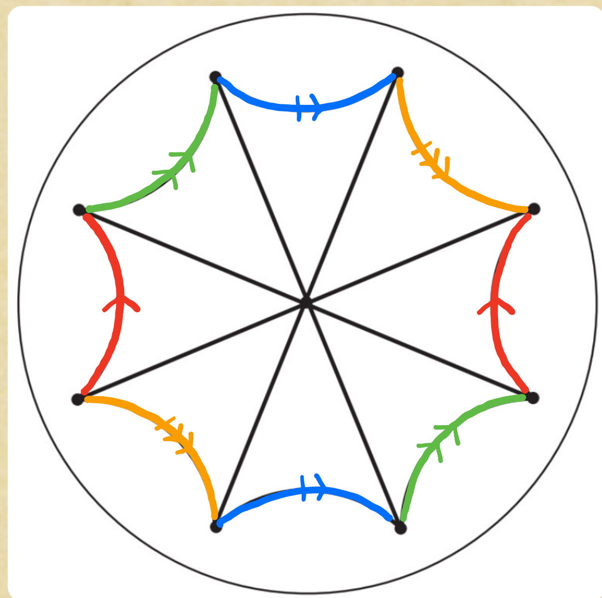
The surface of genus two

Because $\measuredangle P_2 = \pi/8$, the 16 triangles fit together to form a polygon. Because $g \perp h$, the triangles will



fit together in pairs, giving us the desired octagon. ◇

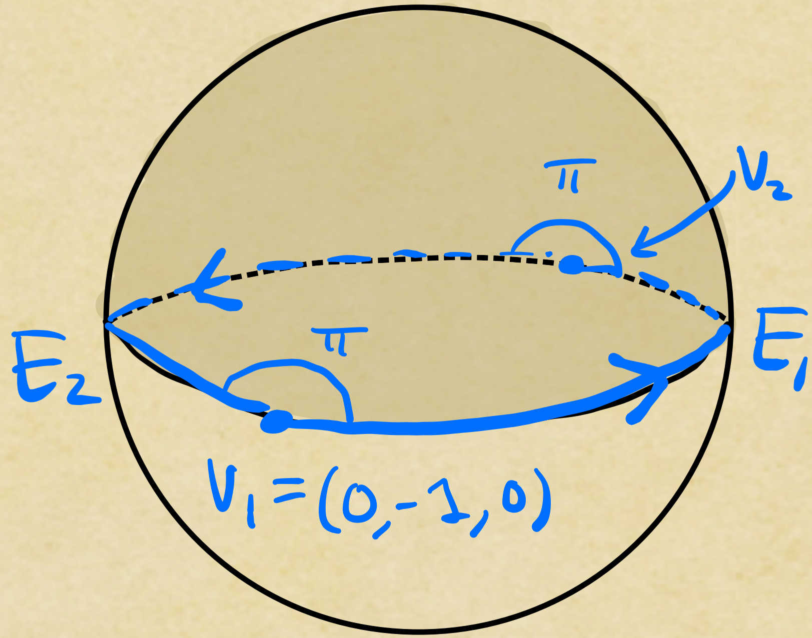
We can now use the edge gluing from before to get a hyperbolic metric on Σ_2.



The projective plane

In S^2 we may consider the polygon

$$X = \{ z \geq 0 \}.$$



Namely, we define edges

$$E_1 = \{ z = 0 \text{ \& } x \geq 0 \}, \; E_2 = \{ z = 0 \text{ \& } x \leq 0 \}.$$

We can then define an edge gluing $\varphi_1 : E_1 \longrightarrow E_2$
$$p \longmapsto -p.$$

The vertices $\{V_1, V_2\} = E_1 \cap E_2$ have

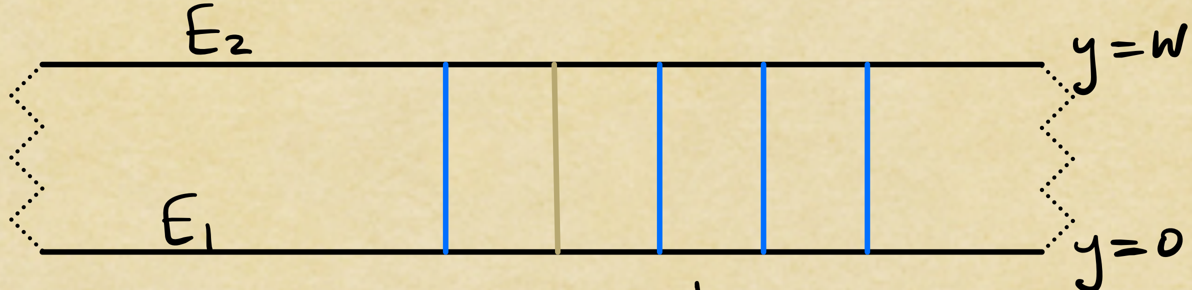
$$\measuredangle(V_1) = \pi \quad \text{\&} \quad \measuredangle(V_2) = \pi ,$$

So this edge gluing determines a <u>spherical surface</u> Known as the <u>projective plane</u>.

Gluing Euclidean strips

In $(\mathbb{R}^2, d_{euc})$, the region between a pair of parallel lines is a non-compact polygon.

Given any such strip X, we can apply an isometry to turn X into



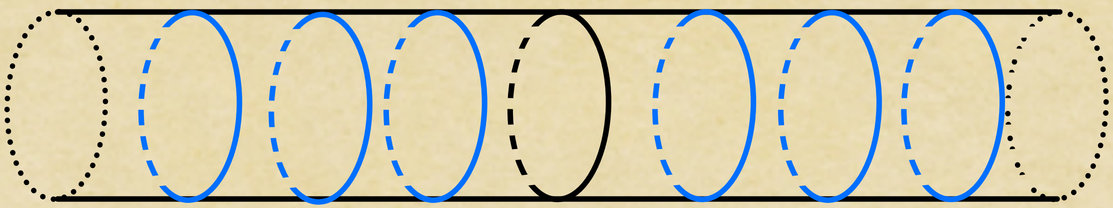
, for some $w>0$. Setting $E_1 = \{y=0\}$ & $E_2 = \{y=w\}$, we can consider the gluings

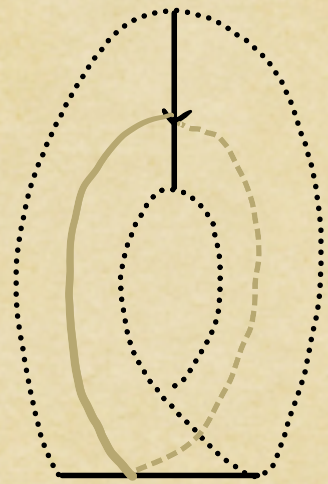
$$\varphi_1 : E_1 \to E_2 \qquad \text{OR} \qquad \psi_1 : E_1 \to E_2$$
$$(x,0) \mapsto (x,w) \qquad\qquad (x,0) \mapsto (-x,w).$$

(We make no claim of uniqueness at this stage.)

Gluing Euclidean strips

These gluings give Euclidean surfaces (there are no vertices to check) homeomorphic to the (infinite) cylinder & Möbius strip, respectively.

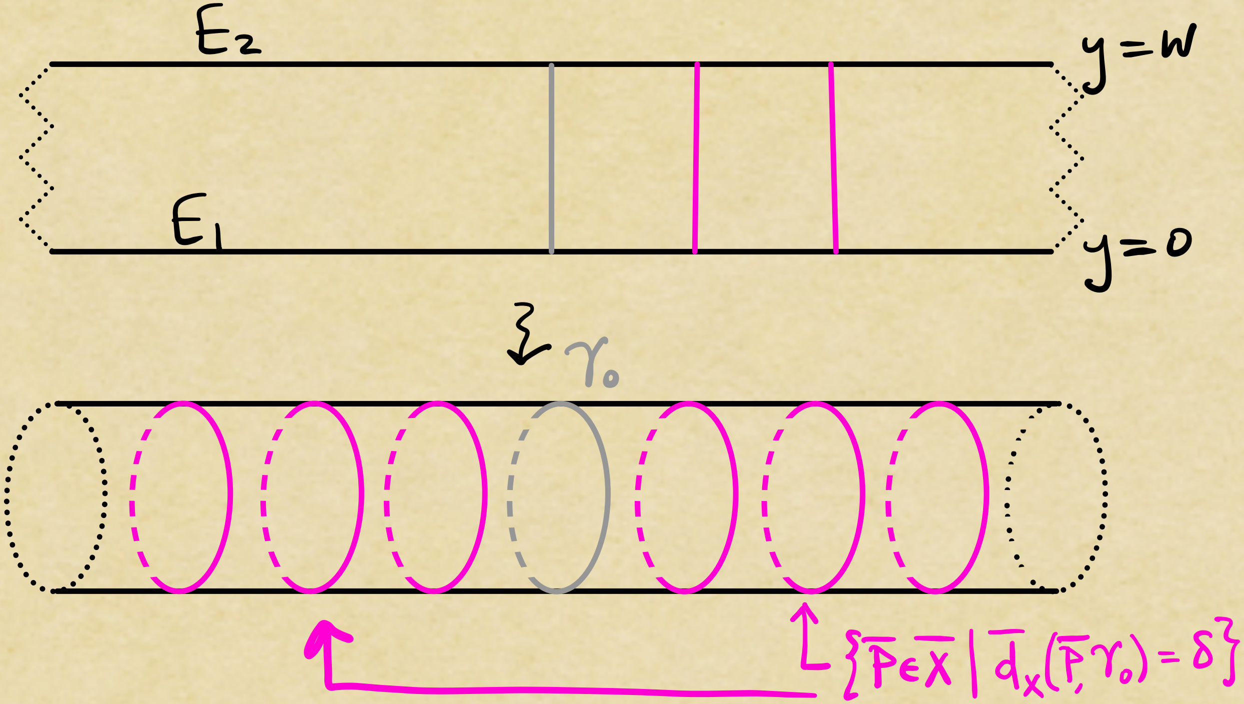




Each of these is foliated by a family of closed geodesics. On the cylinder, these all have length w; on the Möbius strip we see "period doubling."

Gluing Euclidean strips

It's important to note that Euclidean cylinders have **constant width.**



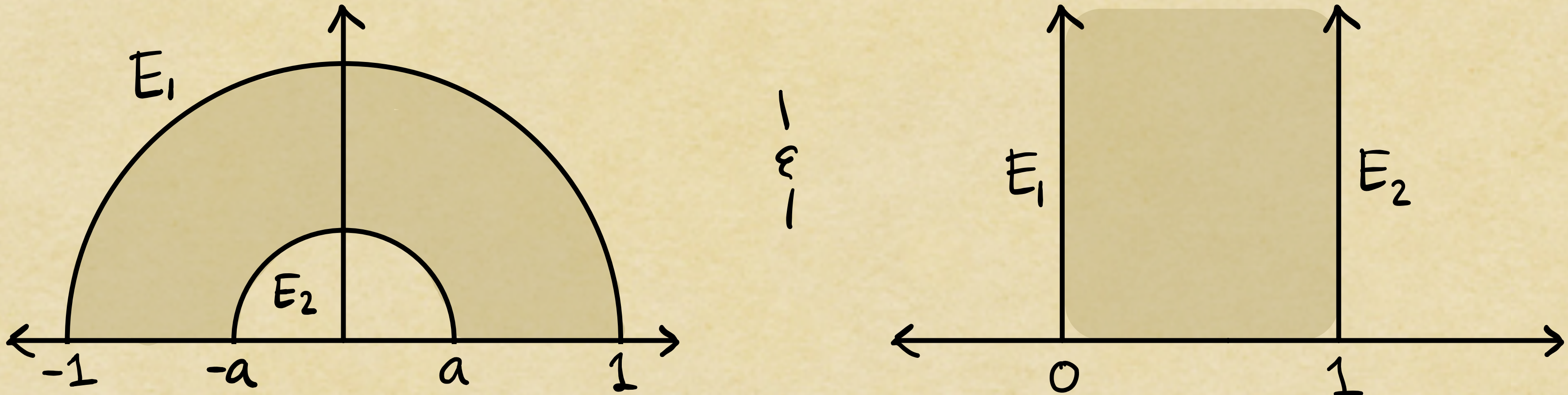
Namely, if we fix a closed geodesic γ_0 in $\bar{X}$, then

$$\{\bar{p}\in\bar{X} \mid \bar{d}_X(\bar{p},\gamma_0)=\delta\}$$

is a pair of closed curves of length <u>w</u>, for every $\delta>0$.

Gluing hyperbolic strips

In $(\mathbb{H}^2, d_{hyp})$ we must consider two types of strip, up to isometry:



$$X = \{ z \mid a \leq |z| \leq 1 \} \qquad X = \{ z \mid 0 \leq \operatorname{Re} z \leq 1 \}$$

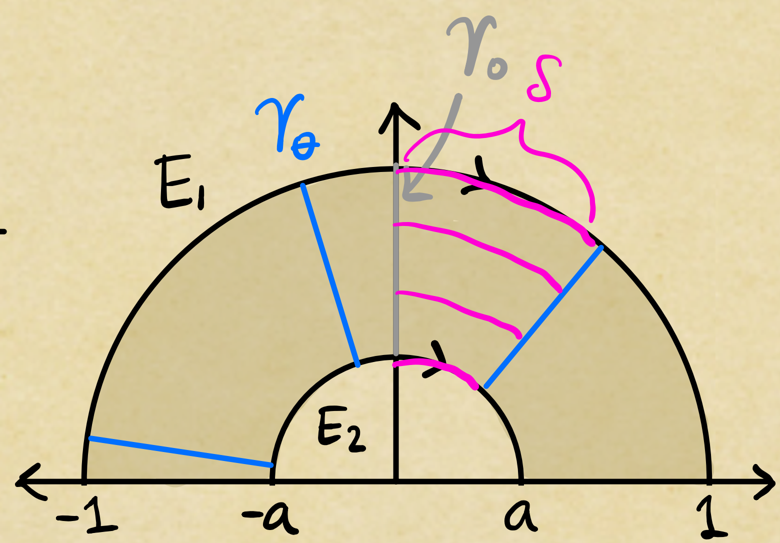
$$E_1 = \{ z \mid |z| = 1 \} \qquad E_1 = \{ z \mid \operatorname{Re} z = 0 \}$$

$$E_2 = \{ z \mid |z| = a \} \qquad E_2 = \{ z \mid \operatorname{Re} z = 1 \}$$

Let's use these to build <u>hyperbolic cylinders</u>.

Gluing hyperbolic strips

Case 1: Distinct endpoints



Consider the gluing

$$\varphi_1 : E_1 \longrightarrow E_2$$
$$z \longmapsto az.$$

This is an isometry, since it's a homothety.
The usual theorems then give us a hyperbolic surface.

However, this hyperbolic cylinder does NOT have constant width. For each $\theta \in (-\pi/2, \pi/2)$, let γ_θ be the closed curve in $\overline{X}$ given by $\gamma_\theta = \{ \overline{t e^{i(\pi/2-\theta)}} \mid t \in [a,1] \} \subset \overline{X}$.

Check: (1) $\ell_{hyp}(\gamma_\theta) = -\ln(a) \cdot \sec\theta$

(2) $\overline{P} \in \gamma_\theta \Rightarrow d_{\overline{X}}(\overline{P}, \gamma_0) = \ln(\sec\theta + \tan|\theta|)$.

Gluing hyperbolic strips

Check: (1) $\ell_{hyp}(\gamma_\theta) = -\ln(a)\cdot\sec\theta$

(2) $\overline{P}\in\gamma_\theta \Rightarrow \overline{d}_X(\overline{P},\gamma_0) = \ln(\sec\theta + \tan|\theta|)$.

Upshot: $\{\overline{P}\in\overline{X} \mid \overline{d}_X(\overline{P},\gamma_0)=\delta\} = \gamma_{\operatorname{arcsec}(\cosh\delta)}$

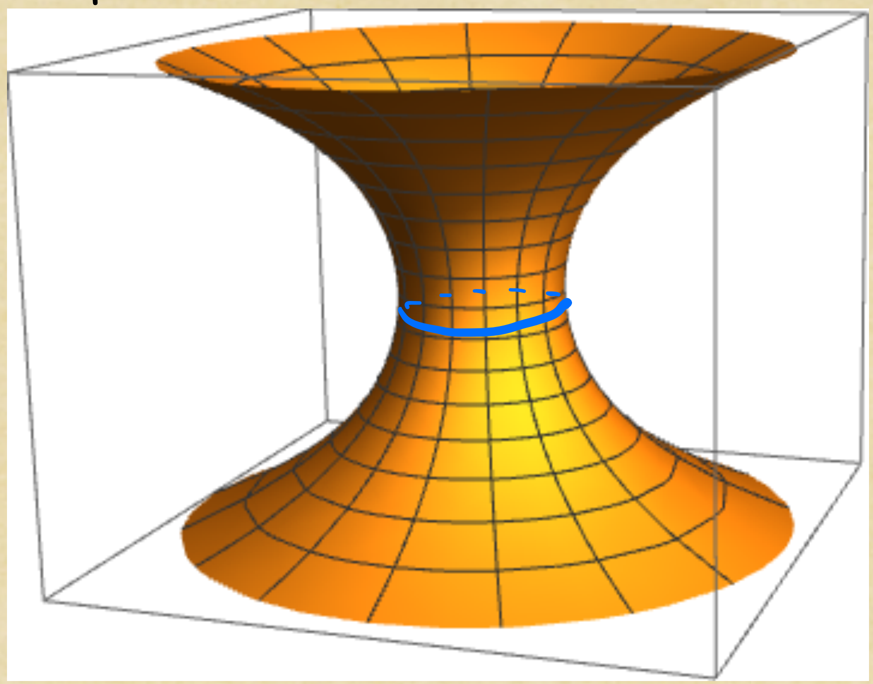
So $\{\overline{P}\in\overline{X} \mid \overline{d}_X(\overline{P},\gamma_0)=\delta\}$ is a pair of closed curves

with hyperbolic length

$\ell_{hyp}(\gamma_{\operatorname{arcsec}(\cosh\delta)})$

$= -\ln(a)\cdot\sec(\operatorname{arcsec}(\cosh\delta))$

$= -\ln(a)\cdot\cosh\delta$



(Not an isometric embedding.)

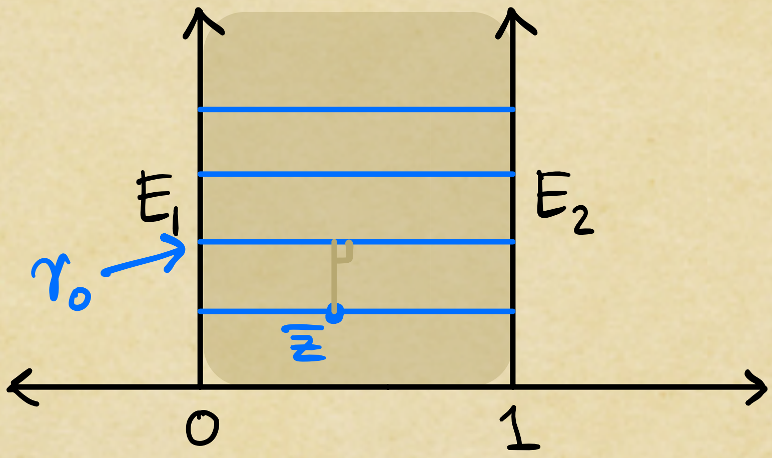
Since $\cosh\delta = \frac{1}{2}(e^{\delta}+e^{-\delta})$, $\overline{X}$ looks like a cylinder with exponentially-expanding width.

Gluing hyperbolic strips

Case (2): A shared endpoint

Consider the gluing

$$\varphi_1 : E_1 \longrightarrow E_2$$
$$z \longmapsto z+1 .$$



Once again, the quotient metric space $(\overline{X}, \bar{d}_X)$ is a hyperbolic cylinder. For all $t \in \mathbb{R}$ we may consider

$$\gamma_t = \{\bar{z} \in \overline{X} \mid \operatorname{Im} z = e^t\} \subset \overline{X}.$$

Then $\ell_{hyp}(\gamma_t) = \int_0^1 \frac{1}{e^t}\, du = e^{-t}$

& $\bar{z} \in \gamma_t \Rightarrow \bar{d}_X(\bar{z}, \gamma_0) = \int_{e^t}^{1} \frac{\sqrt{0^2+1^2}}{y}\, dy = \ln(y)\Big|_{e^t}^{1}$

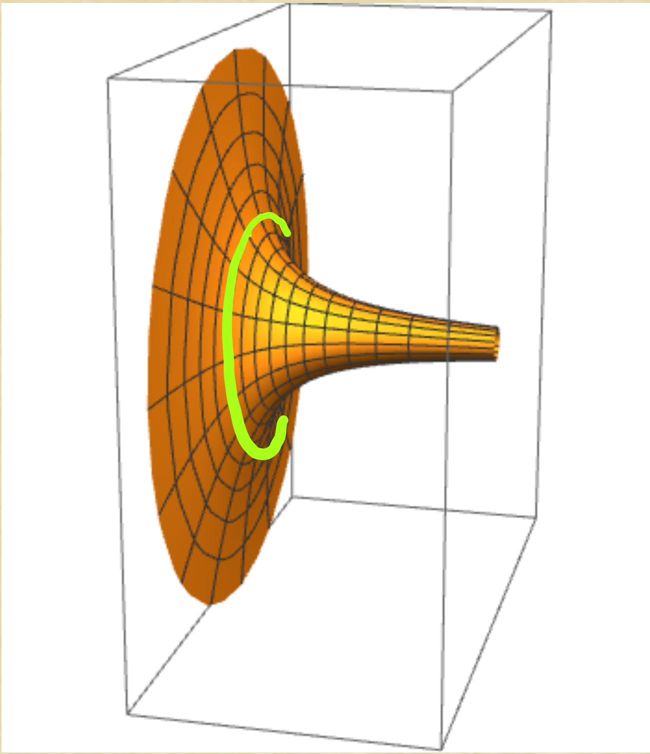
$$= |\ln(e^t)| = |t|$$

Gluing hyperbolic strips

Then $\ell_{hyp}(\gamma_t) = e^{-t}$ & $\bar{z} \in \gamma_t \Rightarrow \bar{d}_X(\bar{z}, \gamma_0) = |t|$, so $\{\bar{z} \in \bar{X} \mid \bar{d}_X(\bar{z}, \gamma_0) = \delta\} = \gamma_t \sqcup \gamma_{-t}$ is a pair of closed curves of hyperbolic length e^{-t} & e^{t}, respectively.

So $\bar{X}$ is a cylinder whose width at one end grows exponentially and at the other end decays exponentially.

The (isometry type of) the image of $\{z \in X \mid \operatorname{Im} z \geq \frac{1}{2\pi}\} \subset X$ in $\bar{X}$ is known as a pseudosphere.



(Not an isometric embedding.)

<u>Next</u>

Still more examples!