Math 4803 February 12, 2024

LAST TIME

Euclidean, hyperbolic, and spherical
surfaces as quotients of polygons.

Key technical requirement: good angle sums.

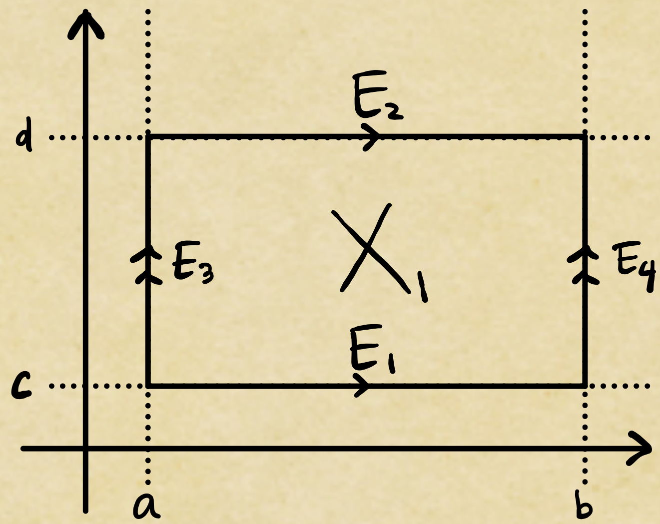
TODAY

Several Euclidean examples, plus
the def'n of homeomorphism.

A torus from a rectangle

Consider the Euclidean rectangle

$$X_1 = [a,b] \times [c,d] \subset (\mathbb{R}^2, d_{euc}).$$



The decorations shown determine an edge gluing with

$$\varphi_1 : E_1 \longrightarrow E_2 \quad \& \quad \varphi_2 : E_3 \longrightarrow E_4$$
$$(x,c) \mapsto (x,d) \qquad\qquad (a,y) \mapsto (b,y)$$

Notice that these are isometries, and that

$$\overline{(a,c)} = \{(a,c), (a,d), (b,d), (b,c)\}.$$

So $\measuredangle(\overline{(a,c)}) = \frac{\pi}{2} + \frac{\pi}{2} + \frac{\pi}{2} + \frac{\pi}{2} = 2\pi$,

and the gluing gives us a Euclidean surface.

Homeomorphisms

A **homeomorphism** between metric spaces (X,d) and (X',d') is a bijection $\varphi: X \to X'$ s.t. both φ & φ^{-1} are continuous. i.e., $\forall\, P \in X, P' \in X'$, & $\varepsilon > 0$ $\exists\, \delta > 0$ s.t. $d(P,Q) < \delta \Rightarrow d'(\varphi(P), \varphi(Q)) < \varepsilon$

& $d'(P', Q') < \delta \Rightarrow d(\varphi^{-1}(P'), \varphi^{-1}(Q')) < \varepsilon$.

Note: isometry $\overset{\Longrightarrow}{\cancel{\Longleftarrow}}$ homeomorphism

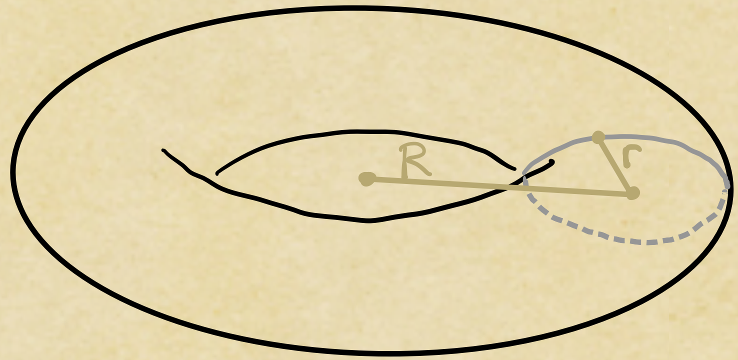
We'll typically (after today) be pretty informal about homeomorphisms, thinking of them as "bijections which may stretch the metric." The key is that they preserve the notions of limits and continuity.

It really was a torus

The 2-dimensional torus, denoted T^2, is the surface in $\mathbb{R}^3$ obtained by revolving about the z-axis the circle

$$(x-R)^2 + z^2 = r^2$$

in the xz-plane, where $R > r$.



Check: Different choices of $R > r$ give homeomorphic subsets of $\mathbb{R}^3$.

Lemma Let $(\overline{X}_1, \overline{d}_{X_1})$ be the quotient metric space obtained from $X_1 = [a,b] \times [c,d]$ by gluing together opposite edges by Euclidean translations. Then $(\overline{X}_1, \overline{d}_{X_1})$ is homeomorphic to $(T^2, d_{eucl}|_{T^2 \times T^2})$.

It really was a torus

(Proof) First, the precise choices of $b>a$ & $d>c$ are irrelevant:

Check: $[a,b]\times[c,d] \longrightarrow [-\pi,\pi]\times[-\pi,\pi]$

$$(x,y) \longmapsto \left(\frac{2\pi}{b-a}\left(x-\frac{a+b}{2}\right), \frac{2\pi}{d-c}\left(y-\frac{c+d}{2}\right)\right)$$

is a homeomorphism.

So we'll assume that $X_1 = [-\pi,\pi]\times[-\pi,\pi]$.

Now consider $\rho : [-\pi,\pi]\times[-\pi,\pi] \longrightarrow T^2$ defined by

$$\rho(\theta,\phi) = ((R + r\cos\phi)\cos\theta, (R + r\cos\phi)\sin\theta, r\sin\phi).$$

Notice that $\rho(\theta,\phi) = \rho(\theta',\phi')$ iff $\underline{(\theta,\phi)\sim(\theta',\phi')}$, so we can define a $\underline{\text{bijection}}$ $\overline{\rho} : \overline{X_1} \longrightarrow T^2$ by

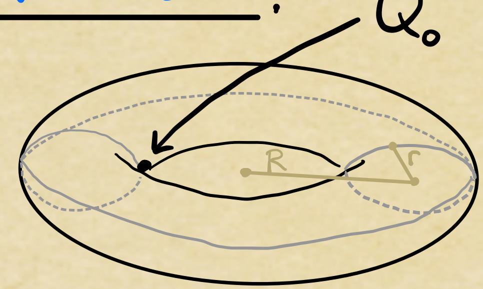
$$\overline{\rho}(\overline{P}) := \rho(P) \text{ for every } \overline{P} \in \overline{X_1}.$$

It really was a torus

Trig/calculus: $\rho: X_1 \to T^2$ is continuous.

Earlier we showed that $\overline{d}(\overline{P}, \overline{Q})$ $\underline{\le d(P, Q)}$, for any partition of (X, d). This can be used to check that $\underline{\overline{\rho} \text{ is also continuous}}$.

Finally, we need to verify that $\underline{\overline{\rho}^{-1} \text{ is also continuous}}$.



We'll verify continuity at

$$Q_0 := \underline{\overline{\rho}\big((\pm\pi, \pm\pi)\big)} = (-R+r, 0, 0).$$

For $Q = (x, y, z) \in T^2$ near Q_0,

$$\Phi^{-1}(x,y,z) = \begin{cases} \pi - \arcsin\left(\frac{z}{r}\right), & z > 0 \\ \pm\pi, & z = 0 \\ -\pi - \arcsin\left(\frac{z}{r}\right), & z < 0 \end{cases}$$

$$\Theta^{-1}(x,y,z) = \begin{cases} \pi - \arcsin\left(\frac{y}{R + r\cos\phi}\right), & y > 0 \\ \pm\pi, & y = 0 \\ -\pi - \arcsin\left(\frac{y}{R + r\cos\phi}\right), & y < 0 \end{cases}.$$

(We could turn this into a formula for $\rho^{-1}(x,y,z)$, but... gross.)

It really was a torus

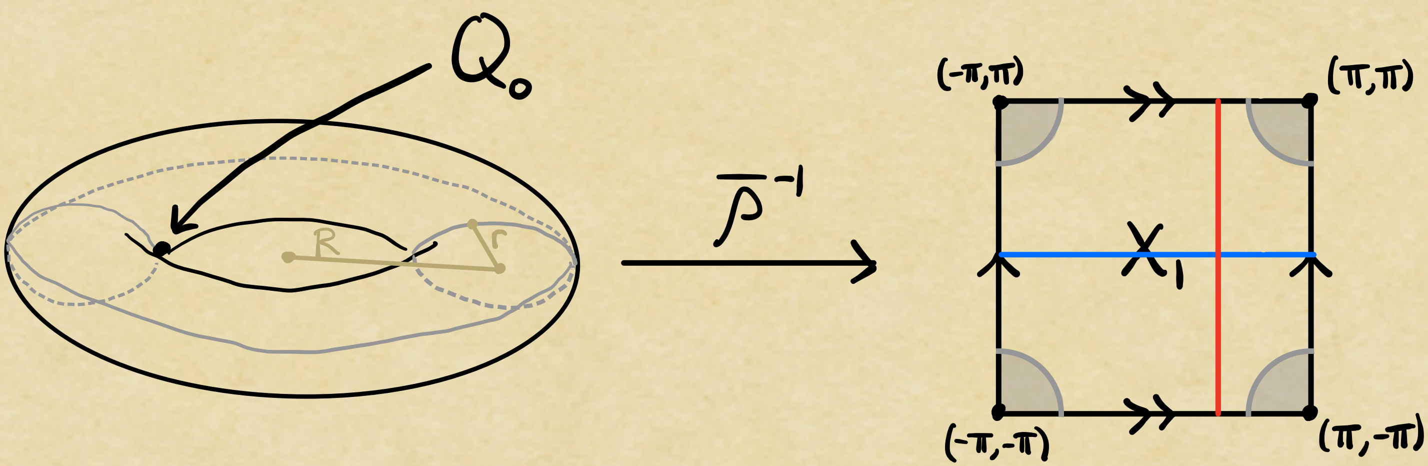
No bar!

By the continuity of $\arcsin : [-1,1] \to [-\pi/2, \pi/2]$, ρ^{-1} will send points close to Q_0 to points near one of

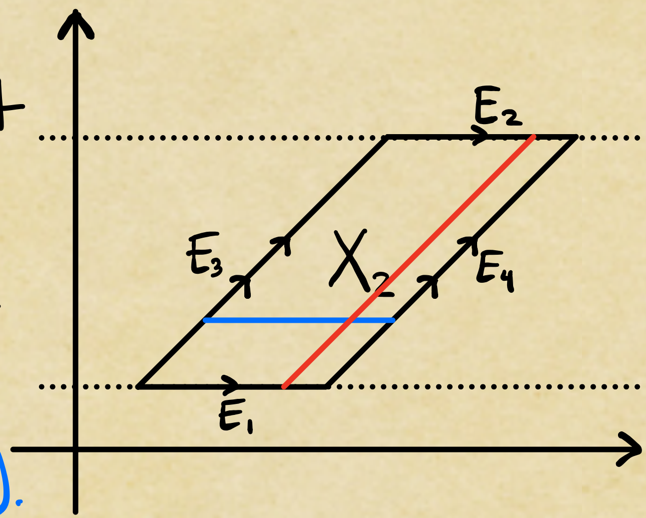
$(-\pi,-\pi)$, $(-\pi,\pi)$, $(\pi,-\pi)$, or (π,π).

It follows that $\bar{\rho}^{-1} : T^2 \to \overline{X_1}$ will send points near Q_0 to points near $(\pm\pi,\pm\pi)$, so $\bar{\rho}^{-1}$ is cts @ Q_0.

Points other than Q_0 are similar, but with fewer cases. ◇



Tori from parallelograms



Lemma Let $(\overline{X}_2, \overline{d}_{X_2})$ be the quotient metric space obtained from a parallelogram X_2 in $(\mathbb{R}^2, d_{euc})$ by gluing together opposite edges by Euclidean translations. Then $(\overline{X}_2, \overline{d}_{X_2})$ is homeomorphic to (T^2, d_{euc}).

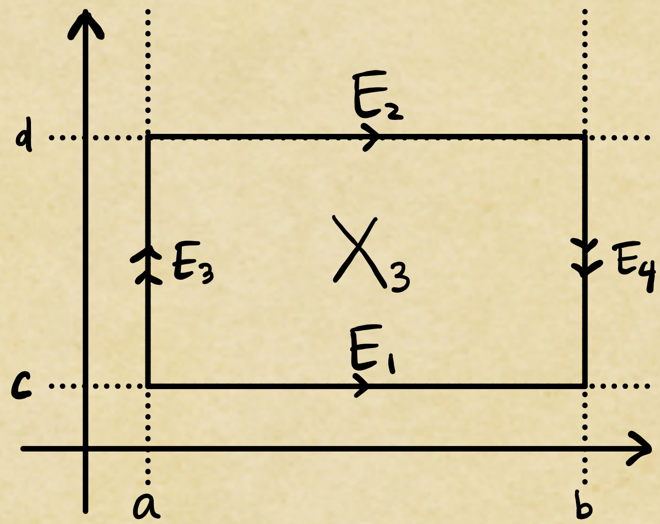
(Proof) On HW 3 you'll construct a homeomorphism $\varphi : X_2 \to X_1$, where $X_1 = [a,b] \times [c,d]$. Moreover,

$$P \sim Q \quad \text{iff} \quad \varphi(P) \sim \varphi(Q), \quad \forall\, P, Q \in X_2,$$

leading to a bijection $\overline{\varphi} : \overline{X_2} \to \overline{X_1}$. Finally, we can check that $\overline{\varphi}$ is a homeomorphism, meaning that $(\overline{X}_2, \overline{d}_{X_2}) \simeq (\overline{X}_1, \overline{d}_{X_2}) \simeq (T^2, d_{euc}|_{T^2 \times T^2})$. $\diamond$

Klein bottles

By using a different edge gluing on the rectangle $[a,b]\times[c,d]$, we obtain a different Euclidean surface.



Let E_1, E_2, E_3, E_4 be as before and consider $\varphi_1 : E_1 \to E_2$ & $\varphi_3 : E_3 \to E_4$

$$(x,c) \mapsto (x,d) \qquad (a,y) \mapsto (b, d-(y-c)).$$

These are both isometries, and

$$\overline{(a,c)} = \{(a,c),(a,d),(b,d),(b,c)\}.$$

Since $\measuredangle(a,c) + \measuredangle(a,d) + \measuredangle(b,d) + \measuredangle(b,c) = 2\pi$

the result is a Euclidean surface.

Klein bottles