# Geometric flows and the geometrization conjecture

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**Abstract**

We give an exposition of two geometric flows — the Ricci flow and the curve-shortening flow — in the context of their application to the geometrization conjecture and the corollary Poincaré conjecture. Our discussion of the Ricci flow will contain high-level intuition, whereas our discussion of the related but simpler curve-shortening flow will formally prove a kind of one-dimensional analog of the Poincaré conjecture.

## 1 Introduction

The geometrization conjecture is a crowning achievement of low-dimensional topology. Posed by Thurston in 1982, it proposes a complete characterization of three-dimensional manifolds by the geometries of their components. The proof of this conjecture by Perelman in 2003 also resolved as a special case the Poincaré conjecture. This conjecture, posed by Poincaré in 1904, proposes a sufficient condition for a three-dimensional manifold to be topologically equivalent to a three-dimensional sphere. The proof of these conjectures hinges on Hamilton's Ricci flow technique, which we aim to demystify in this paper. However, to fully appreciate the significance of the proof, and to motivate a study of geometric flows, we must first understand the history of the conjectures.

## 2 The Conjectures[1]

Of his more than five hundred papers, only six of French mathematician Henri Poincaré's papers dealt with topology. Despite this, these six papers, published between 1895 and 1904, made significant advancements in the field. Poincaré was particularly concerned with three-dimensional manifolds (concisely, *three-manifolds*), as these modeled possible shapes of our universe. This research involved finding sets of invariants that could distinguish different manifolds. In his final topological paper, Poincaré put forth the following question: *"Is it possible for the fundamental group of V [a manifold] to reduce to the identity without V being simply connected?"* [Po]. This question became known as the Poincaré conjecture, which we now state in simpler words.

**Conjecture 2.1** (Poincaré)**.** *If every loop on a closed three-manifold can be continuously deformed to a point, then the manifold is topologically a three-sphere.*

By *loop*, we mean a closed curve, and by *continuously deformed*, we mean sliding the points of the curve along the manifold. A manifold with this condition of being able to contract loops to points is called *simply connected*. While seemingly irrelevant at first glance, simply connectedness is a reasonable condition to request of a three-manifold to be a three-sphere, for in two dimensions, this is a fairly believable fact: Consider wrapping a two-dimensional rubber band around a sphere. The tendency for this rubber band is to want to contract, and on a sphere, it can always contract to a point. However, the same rubber band wrapped around the tube of a torus is trapped from doing so by the tube, so a torus cannot be a sphere (see Figure 1). If a surface contains no such traps, then it is reasonable to intuit that it must be a sphere.

[1]The main source for this section is [O].

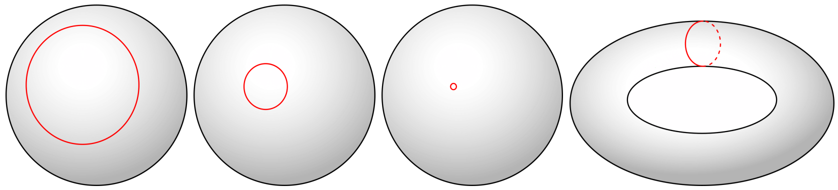

Figure 1: A curve contracting to a point on a sphere vs. a curve trapped on a torus

Poincaré was unsuccessful in proving his conjecture in his lifetime. In the 20th century, numerous incorrect proofs were announced, including from Max Dehn (1908), J. H. C. Whitehead (1934), R. H. Bing (1958), Colin Rourke (1986), and Poénaru (1995), earning the conjecture its formidable status. In 1962, Stephen Smale proved the analogous conjecture for five dimensions and higher, and in 1982, Michael Freedman proved the case of dimension four. However, three dimensions remained an open question.

In the 1970s, geometry saw a resurgence at the hand of American mathematician Bill Thurston, and with it a renewed interest in the conjecture. Thurston sought a complete classification of three-manifolds by their geometry (a notion we will not make precise). He put forth the geometrization conjecture, saying that any three-dimensional manifold can be decomposed into prime (i.e., non-decomposable) manifolds that each admits one of eight *Thurston geometries*. Suffice to say, this is a deep topic deserving of its own paper; we simply remark here that by proving the geometrization conjecture, one also proves the Poincaré conjecture, for it can be shown that any simply connected three-manifold with spherical geometry is topologically equivalent to a three-sphere. Previously, the only reason to believe that the Poincaré conjecture might be true was because nobody could produce a counterexample, but after Thuston, perhaps it was.

## 3 Ricci Flow

In 1982, Richard Hamilton put forth a tool called the *Ricci flow* with the aim of applying it to the geometrization conjecture. The Ricci flow is a *geometric flow* defined on a Riemannian manifold whose purpose is to smooth out regions of extreme curvature. Loosely, a geometric flow is a process for changing a geometric object over time. This is codified by a partial differential equation, and the particular PDE that codifies the Ricci flow is

$$\frac{d}{dt} g_{ij}(t) = -2R_{ij} \tag{3.1}$$

where $g_{ij}$ is the metric tensor and $R_{ij}$ is the Ricci curvature tensor of the manifold (see [S] for a formal study of these objects). The hope is that by evolving a manifold via Ricci flow, regions of the manifold might admit one of Thurston's geometries.

There are two perspectives for the Ricci flow: Ricci flow as a physical force and Ricci flow as a heat equation. As a force, we envision the Ricci flow pushing on a manifold and deforming it in space. Suppose aliens were to wrap the Earth in a large sheet of rubber to obtain a smooth approximation of the surface. By applying the Ricci flow to the rubber surface — and for a suitable definition of curvature — high-curvature regions like mountains and valleys would be smoothed down and low-curvature regions like the American Great Plains would not budge. The net effect would be the rubber surface becoming simpler.

Alternatively, Ricci flow as a heat equation requires us to think about the metric tensor as being imposed onto a manifold. Typically, we draw surfaces in some ambient $\mathbb{R}^3$ space and encapsulate the inherited inner product into the metric tensor $g_{ij}$ to take measurements. However, we could just as well declare our own metric tensor on a surface (or manifold) and speak nothing of an ambient space. For instance, one could declare that all points on a sphere be one unit away from each other; such a sphere could not be realized in $\mathbb{R}^3$, but it is well-defined mathematically. In this way, we envision curvature as temperature on a stationary manifold and Ricci flow as an equation for dissipating heat over the manifold.

By the early 1990s, Hamilton and his collaborators had proved that compact surfaces evolved under Ricci flow wound up with constant curvature. However, up one dimension, three-manifolds were susceptible to developing *singularities* — places where the metric ceased to be a metric. Such singularities were catastrophic and appeared unavoidable and intractable. The last major breakthrough came from Russian mathematician Grisha Perelman who, in a series of three papers published to the arXive in 2002 and 2003, laid the foundation for a technique for resolving singularities by surgering them out and gluing back on spherical caps. This is known as Ricci flow with surgery. After sufficient scrutiny from the mathematical community, Perelman's results were accepted and the geometrization and Poincaré conjectures proved.

# 4 Curve-Shortening Flow

Equipped with an intuition for Ricci flow, we now discuss some mathematics of the related but more approachable curve-shortening flow (CSF). CSF was employed by Perelman in [P, §2] to tie up a loose end related to the geometrization conjecture, but we will not remark further on that here. Rather, our interest in CSF is restricted to its usefulness as a toy model for the mathematics of Ricci flow.

## 4.1 Introduction to CSF

Let $\gamma(t)$ be a family of curves. We think of the parameter $t$ to $\gamma$ as time, and by increasing this parameter, $\gamma$ plays a movie of the initial curve $\gamma(0)$ evolving over time (see Figure 2). Let $\vec{\alpha}(t,s)$ be a parametrization of the curve $\gamma(t)$ with arc length parameter $s$. Denote $\vec{n}(t,s)$ the planar normal of $\vec{\alpha}(t,s)$ and $k(t,s)$ the planar curvature of $\vec{\alpha}(t,s)$.

**Definition 4.1.** The **curve-shortening flow** is a geometric flow on a planar curve that is defined by the differential equation

$$\frac{\partial}{\partial t}\vec{\alpha}(t,s) = k(t,s)\vec{n}(t,s) \tag{4.2}$$

or more concisely

$$\vec{\alpha}_t = k\vec{n},$$

where the subscript denotes a partial derivative with respect to $t$. In words, this says that each point on a curve $\vec{\alpha}$ moves in the direction of its normal by the amount of its curvature.

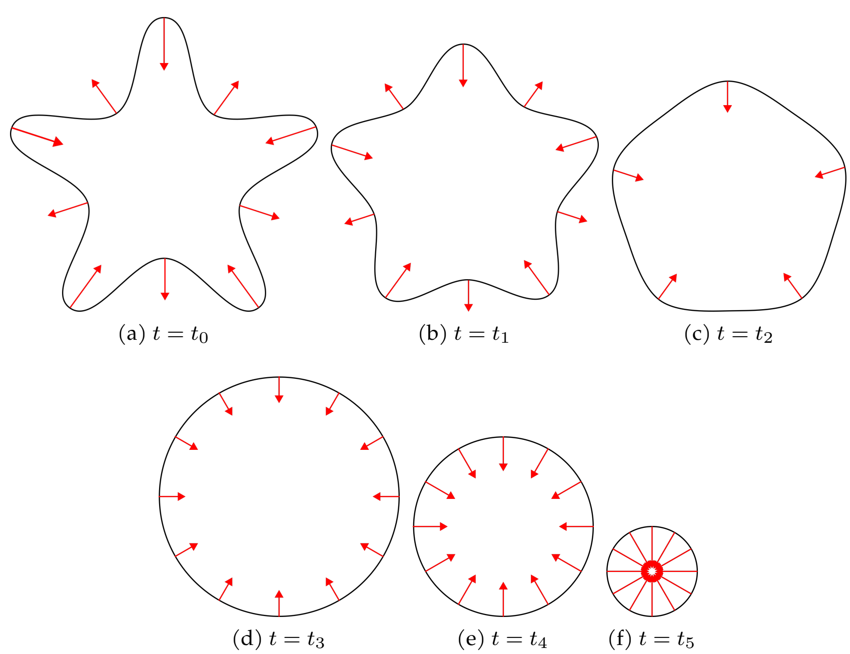


Figure 2: Sketch of the evolution of a closed curve under CSF

**Example 4.3.** Classmates may recognize (4.2) as the derivative of (34) in [W, §4.2], which was used to prove that a surface curve which minimizes length between two points is a geodesic. Indeed, this proof employed CSF.

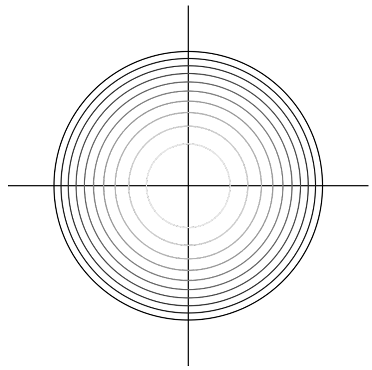

Figure 3: The evolution of the unit circle under CSF at equal times between 0 and 0.5

**Example 4.4** (Due to [Ga])**.** We consider the simple case where $\gamma(0)$ is the unit circle centered at the origin parametrized by $\vec{\alpha}(0,\theta) = (\cos\theta, \sin\theta)$. This has curvature $k \equiv 1$ and unit normal $\vec{n}(0,\theta) = (-\cos\theta, -\sin\theta) = -\vec{\alpha}(0,\theta)$. Due to its radial symmetry, $\vec{\alpha}(0,\theta)$ will remain a circle under CSF but its radius may change. Therefore, its evolution is described by

$$\vec{\alpha}(t,\theta) = r(t)(\cos\theta, \sin\theta)$$

where we need only solve for its radius $r(t)$. Since $\vec{\alpha}(t,\theta)$ is a circle, its curvature is simply $k(t,\theta) = \frac{1}{r(t)}$ and its normal vector will remain as $\vec{n}(t,\theta) = -\vec{\alpha}(0,\theta)$. Thus, applying CSF with (4.2), we have the initial value problem

$$\begin{aligned}\vec{\alpha}(0,\theta) &= (\cos\theta, \sin\theta)\\ \vec{\alpha}_t(t,\theta) &= k(t,\theta)\vec{n}(t,\theta) = -\frac{1}{r(t)}(\cos\theta, \sin\theta),\end{aligned}$$

although we are only concerned with

$$\begin{aligned}r(0) &= 1\\ r'(t) &= -\frac{1}{r(t)}.\end{aligned}$$

Solving for $r(t)$ yields

$$r(t) = \sqrt{1-2t},$$

so the final evolution equation for $\gamma$ is

$$\vec{\alpha}(t,\theta) = \sqrt{1-2t}(\cos\theta, \sin\theta).$$

We observe from this that the radius of the unit circle $\gamma(0)$ decreases to 0 at $t = 0.5$, whereupon $\vec{\alpha}_t$ is no longer defined and the flow stops (see Figure 3).

With an example under our belt, it is natural to next poke at the behavior of the CSF in the same way that Hamilton and Perelman poked at the behavior of Ricci flow; we might ask questions such as, *When evolved under the CSF...*

1. *do simple closed curves approach a circle? A point?*
2. *do* (*non*)*convex simple closed curves remain* (*non*)*convex?*
3. *do simple closed curves develop singularities? How long does it take for those singularities to develop?*

Each of these questions was answered in a series of papers written in the 1980s and 1990s. Here, we will get a taste for working with geometric flows by proving one of the earliest results in this line of questioning.

## 4.2 An isoperimetric inequality applied to CSF

In [G], Gage proves that in some sense, a closed convex $C^2$ planar curve becomes more circular when evolved under CSF. Since Ricci flow applied to the Poincaré conjecture attempts to flow three-manifolds to a sphere, then it is not far-fetched to consider this a one-dimensional analog restricted to convex curves. The object that will allow us to make such a claim is the *isoperimetric ratio*.

**Definition 4.5.** The **isoperimetric ratio** of a simple closed curve of length $L$ and enclosed area $A$ is $L^2/A$.

Recall from class that the isoperimetric inequality

$$L^2 \geq 4\pi A$$

says that a simple closed curve of length $L$ maximizes area if and only if it is a circle. Thus, the isoperimetric ratio satisfies

$$\frac{L^2}{A} \geq 4\pi$$

with equality if and only if the curve is a circle. We can think of a curve having small isoperimetric ratio as being more circular, and a curve having large isoperimetric ratio as being less circular. Thus, the isoperimetric ratio is a measuring device for how circular a simple closed curve is! And we will soon prove that it cannot increase under CSF, meaning a curve's "circularity" will either remain constant or increase.

Of course, we make this claim only for *convex* simple closed curves. This is due to our argument's reliance on the fact that convex curves sit entirely on one side of each of their tangent lines. We call such tangent lines *supporting* tangent lines, and our use of this convexity fact is encoded in the *support function* $p$. This function describes the distance from any supporting tangent line to the origin. Its primary use to us is in the proof of the following lemma, which we omit for the sake of space.

**Lemma 4.6.** *If $\vec{\alpha}$ is a closed, convex, $C^1$, piecewise $C^2$ curve which satisfies the inequality*

$$\int_{\vec{\alpha}} p^2 \, ds \leq \frac{LA}{\pi}$$

*for some choice of origin, then the inequality*

$$\pi \frac{L}{A} \leq \int_{\vec{\alpha}} k^2 \, ds$$

*is also satisfied.*

This takes us to our main result.

**Theorem 4.7** (Gage)**.** *Let $\gamma(t)$ be a family of curves that evolves according to the curve-shortening flow. Then the derivative of the isoperimetric ratio is given by*

$$\left(\frac{L^2}{A}\right)_t = -2\frac{L}{A}\left(\int_\gamma k^2 ds - \pi\frac{L}{A}\right).$$

If the right-hand term in parentheses is non-negative, then the change in isoperimetric ratio over time must be non-positive. Indeed, this is what Lemma 4.6 tells us! Thus, the "circularity" of a curve evoled under CSF must either remain the same or increase.

*Proof.* We first compute the time derivative of the isoperimetric inequality via quotient rule:

$$\begin{aligned}\left(\frac{L^2}{A}\right)_t &= \frac{2LL_tA - L^2A_t}{A^2} \\ &= \frac{2LL_t}{A} - \frac{L^2A_t}{A^2} \\ &= -2\frac{L}{A}\left(-L_t + \frac{A_t}{2}\frac{L}{A}\right).\end{aligned}$$

This closely matches what we expect, however it remains to compute $L_t$ and $A_t$.

The proof for $L_t$ follows exactly from our work in class on distance-minimizing surface curves, so we fastforward to arrive at

$$L_t = -\int k^2 \, ds.$$

This solves half of the puzzle.

Since we think of the family of curves $\gamma(t)$ as playing a movie of the original $\gamma(0)$ curve, then we want to use the same parameter for every $\gamma(t)$. However, after evolving $\gamma(0)$, we can no longer assume that its parametrization is unit-speed. Thus, we introduce the non-unit speed parameter $u$ and *arc length element* $\|\partial\vec{\alpha}/\partial u\|$. Now, to compute the derivative of area, we begin with the formula

$$A = -\frac{1}{2}\int \langle \vec{\alpha}, \vec{n}\rangle \left\|\frac{\partial\vec{\alpha}}{\partial u}\right\| du,$$

which is obtainable via Green's theorem. Differentiating via chain rule and product rule yields

$$A_t = -\frac{1}{2}\int \left( \langle \vec{\alpha}_t, \vec{n}\rangle \left\|\frac{\partial\vec{\alpha}}{\partial u}\right\| + \langle \vec{\alpha}, \vec{n}_t\rangle \left\|\frac{\partial\vec{\alpha}}{\partial u}\right\| + \langle \vec{\alpha}, \vec{n}\rangle \left\|\frac{\partial\vec{\alpha}}{\partial u}\right\|_t \right) du.$$

Each of these quantities is in some way known except for $\vec{n}_t$ and $\|\partial\vec{\alpha}/\partial u\|$, so we make two detours.

Beginning with $\vec{n}_t$, we compute the second partial derivatives $\vec{\alpha}_{ut}$ and $\vec{\alpha}_{tu}$. The former is interpreted as a non-unit tangent changing over time, which is equivalently

$$\vec{\alpha}_{ut} = \left( \left\|\frac{\partial\vec{\alpha}}{\partial u}\right\| \vec{t} \right)_t, \tag{4.8}$$

where $\vec{t}$ is the unit tangent vector. The latter is the evolution equation (4.2) differentiated with respect to $u$, which yields

$$\vec{\alpha}_{tu} = (k\vec{n})_u = k_u\vec{n} + k\vec{n}_u. \tag{4.9}$$

By equality of mixed partials, (4.8) and (4.9) are equal. Thus, equating them and taking their inner product with $\vec{n}$ gives

$$\begin{aligned} \left\langle \left( \left\|\frac{\partial\vec{\alpha}}{\partial u}\right\| \vec{t} \right)_t, \vec{n} \right\rangle &= \langle k_u\vec{n} + k\vec{n}_u, \vec{n}\rangle \\ &= k_u, \end{aligned}$$

and by applying our trusty product rule trick to the left-hand side, we get

$$-\left\langle \left\|\frac{\partial\vec{\alpha}}{\partial u}\right\| \vec{t}, \vec{n}_t \right\rangle = k_u$$

or

$$\langle \vec{t}, \vec{n}_t\rangle = -\frac{k_u}{\left\|\frac{\partial\vec{\alpha}}{\partial u}\right\|}.$$

However, since $\vec{n}$ is a unit vector, $\vec{n}_t$ is perpendicular to $\vec{n}$ and thus parallel to $\vec{t}$, so

$$\vec{n}_t = -\frac{k_u}{\left\|\frac{\partial\vec{\alpha}}{\partial u}\right\|}\vec{t}.$$

Our second detour is to compute $\left\|\frac{\partial\vec{\alpha}}{\partial u}\right\|_t$, which in the process of computing $L_t$ is found to be

$$\left\|\frac{\partial\vec{\alpha}}{\partial u}\right\|_t = -k^2 \left\|\frac{\partial\vec{\alpha}}{\partial u}\right\|.$$

Putting the pieces together now,

$$\begin{aligned} A_t &= -\frac{1}{2}\int\left(\langle\vec{\alpha}_t,\vec{n}\rangle\left\|\frac{\partial\vec{\alpha}}{\partial u}\right\| + \left\langle\vec{\alpha}, -\frac{k_u}{\left\|\frac{\partial\vec{\alpha}}{\partial u}\right\|}\vec{t}\right\rangle\left\|\frac{\partial\vec{\alpha}}{\partial u}\right\| - k^2\langle\vec{\alpha},\vec{n}\rangle\left\|\frac{\partial\vec{\alpha}}{\partial u}\right\|\right)du \\ &= -\frac{1}{2}\int\left(\langle\vec{\alpha}_t,\vec{n}\rangle\left\|\frac{\partial\vec{\alpha}}{\partial u}\right\| - k_u\langle\vec{\alpha},\vec{t}\rangle - k^2\langle\vec{\alpha},\vec{n}\rangle\left\|\frac{\partial\vec{\alpha}}{\partial u}\right\|\right)du. \end{aligned}$$

Substituting the evolution equation for $\vec{\alpha}_t$ gives

$$\begin{aligned} A_t &= -\frac{1}{2}\int\left(\langle k\vec{n},\vec{n}\rangle\left\|\frac{\partial\vec{\alpha}}{\partial u}\right\| - k_u\langle\vec{\alpha},\vec{t}\rangle - k^2\langle\vec{\alpha},\vec{n}\rangle\left\|\frac{\partial\vec{\alpha}}{\partial u}\right\|\right)du \\ &= -\frac{1}{2}\int\left(k\left\|\frac{\partial\vec{\alpha}}{\partial u}\right\| - k_u\langle\vec{\alpha},\vec{t}\rangle - k^2\langle\vec{\alpha},\vec{n}\rangle\left\|\frac{\partial\vec{\alpha}}{\partial u}\right\|\right)du \\ &= -\frac{1}{2}\int k\left\|\frac{\partial\vec{\alpha}}{\partial u}\right\|du + \frac{1}{2}\int k_u\langle\vec{\alpha},\vec{t}\rangle\,du + \frac{1}{2}\int k^2\langle\vec{\alpha},\vec{n}\rangle\left\|\frac{\partial\vec{\alpha}}{\partial u}\right\|du. \end{aligned}$$

As it turns out (or in other words, I have no idea what the author did),

$$A_t = -2\pi.$$

Finally, plugging in the values for $L_t$ and $A_t$ achieves our desired result. □

In pursuit of a "one-dimensional Poincaré conjecture" whose proof parallels Perelman's use of Ricci flow, we proved that a convex simple closed curve tends to become more circular when evolved under the curve-shortening flow. Many of our questions about the behavior of CSF still remain, such as whether the curve will encounter flow-breaking singularities along its journey, or whether the curve really becomes a circle or merely approaches one. Readers interested in answers to these questions may consult the following sources:

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