Math 4803

April 8, 2024

SO FAR

Geometry in spaces locally isometric to

$(\mathbb{R}^2, d_{euc})$, (S^2, d_{sph}), and $(\mathbb{H}^2, d_{hyp})$.

TODAY

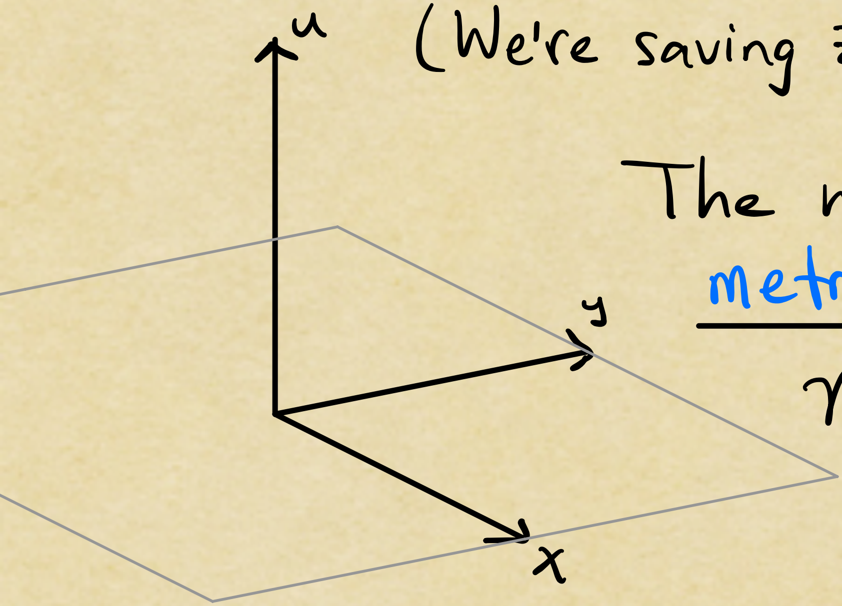
The basic geometry of hyperbolic 3-space.

Definitions & first properties

As a set, 3-dimensional hyperbolic space is

$$\mathbb{H}^3 = \{(x,y,u) \in \mathbb{R}^3 \mid u > 0\}.$$

(We're saving z for $z = x + iy \in \mathbb{C}$.)



The metric d_{hyp} on $\mathbb{H}^3$ will be a path metric, with a diff'able curve

$$\gamma(t) = (x(t), y(t), u(t)), \quad a \le t \le b,$$

having length

$$\ell_{hyp}(\gamma) := \int_a^b \frac{\sqrt{(x'(t))^2 + (y'(t))^2 + (u'(t))^2}}{u}\, dt .$$

Then, for any two points $P, Q \in \mathbb{H}^3$,

$$d_{hyp}(P,Q) := \inf\{\ell_{hyp}(\gamma) \mid P \overset{\gamma}{\rightsquigarrow} Q\}.$$

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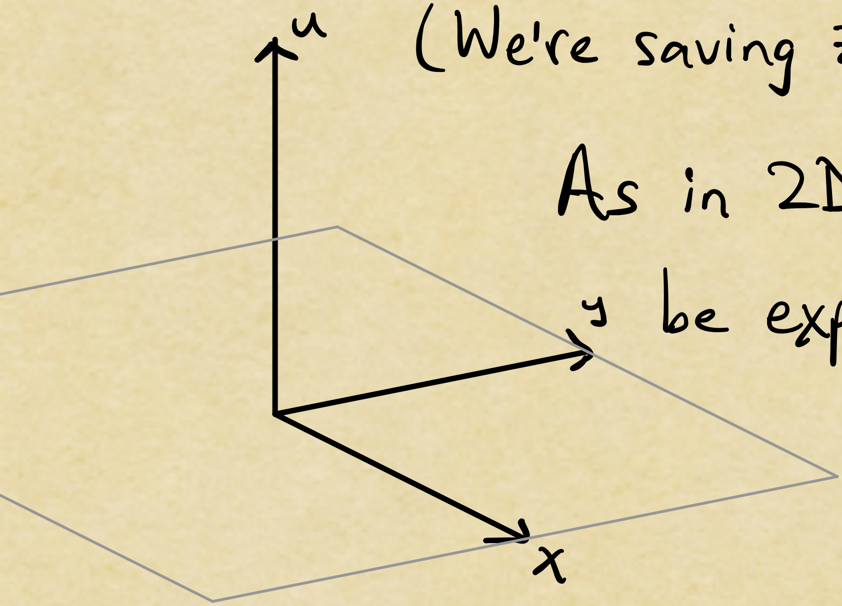


As in 2D, the 3D hyperbolic length can also be expressed via the hyperbolic norm: for a vector $\vec{v} \in \mathbb{R}^3$ based at $(x,y,u) \in \mathbb{H}^3$, we have

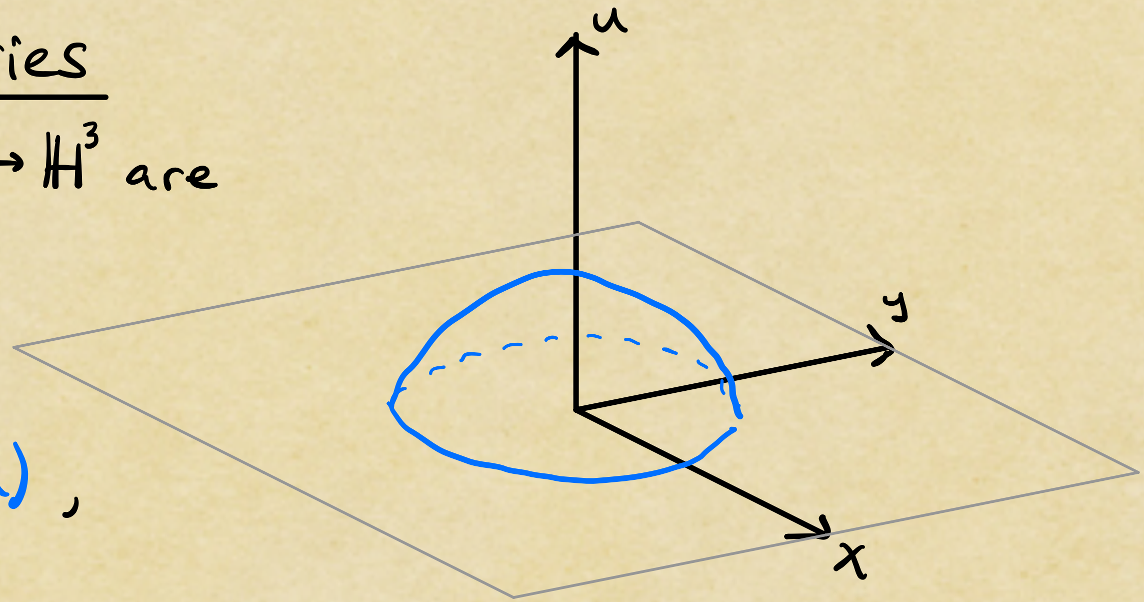
$$\|\vec{v}\|_{hyp} := \frac{1}{u}\|\vec{v}\|_{euc}.$$

Then $\ell_{hyp}(\gamma) = \int_a^b \|\gamma'(t)\|_{hyp}\,dt$.

$\frac{1}{u}(dx^2 + dy^2 + dz^2)$

Some basic isometries

The following maps $\varphi : \mathbb{H}^3 \to \mathbb{H}^3$ are isometries w.r.t. d_{hyp}.



Horizontal translations

$\varphi(x,y,u) = (x+x_0, y+y_0, u)$,

for some $(x_0, y_0) \in \mathbb{R}^2$.

Homothetics

$\varphi(x,y,u) = (\lambda x, \lambda y, \lambda u)$, for some $\lambda > 0$.

Rotations about the u-axis

For some $\theta \in \mathbb{R}$,

$$\varphi(x,y,u) = (x\cos\theta - y\sin\theta,\ x\sin\theta + y\cos\theta,\ u).$$

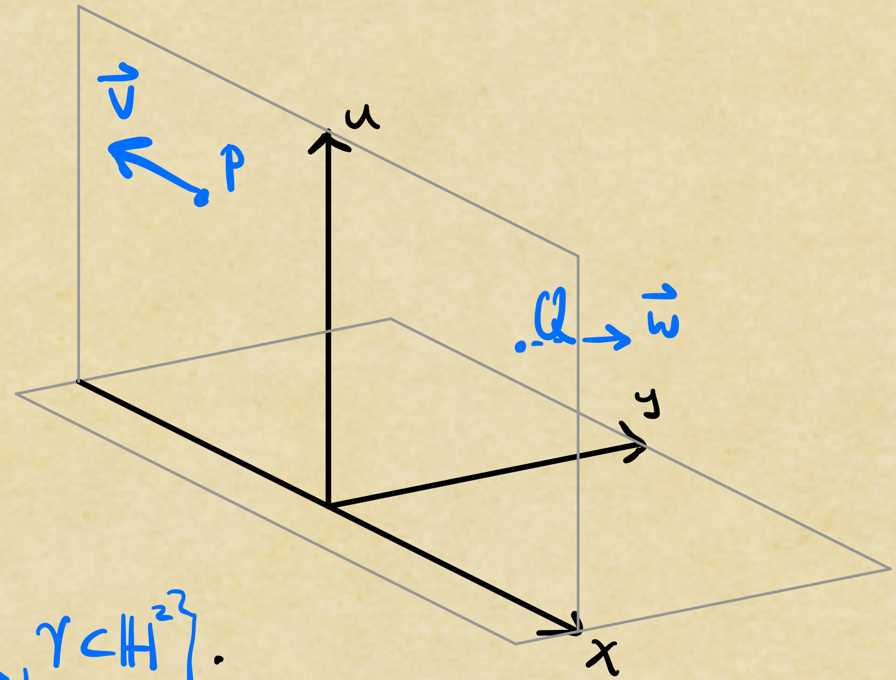
Inversion across unit sphere

$$\varphi(x,y,u) = \left(\frac{x}{x^2+y^2+u^2}, \frac{y}{x^2+y^2+u^2}, \frac{u}{x^2+y^2+u^2}\right).$$

Fundamental properties

$\mathbb{H}^3$ contains an important 2D plane:

$$H = \{(x,0,u) \mid u>0\} \subset \mathbb{H}^3.$$



We can define a metric d_H on H via

$$d_H(P,Q) := \inf\{\ell_{hyp}(\gamma) \mid P \overset{\gamma}{\rightsquigarrow} Q,\ \gamma \subset \mathbb{H}^2\}.$$

Facts: (1) $\forall P,Q \in H$, $d_H(P,Q) = d(P,Q)$.

(2) (H, d_H) is isometric to $(\mathbb{H}^2, d_{hyp})$

(In fact, we'll write $\underline{\mathbb{H}^2 \subset \mathbb{H}^3}$.)

This plane will help us prove:

<u>Thm</u>. The hyperbolic space $(\mathbb{H}^3, d_{hyp})$ is <u>isotropic</u>.

Fundamental properties

<u>Thm</u>. The hyperbolic space $(\mathbb{H}^3, d_{hyp})$ is isotropic.

(Proof.) Given $P, Q \in \mathbb{H}^3$ and $\vec{v}, \vec{w} \in \mathbb{R}^3$ based at P, Q, resp., with $\|\vec{v}\|_{hyp} = \|\vec{w}\|_{hyp}$. We need an isom. $\varphi : \mathbb{H}^3 \to \mathbb{H}^3$ s.t. $\underline{\varphi(P) = Q}$ and $\underline{(D_P\varphi)(\vec{v}) = \vec{w}}$.

Let φ_1 & φ_5^{-1} be <u>horizontal transl.</u> such that $\varphi_1(P), \varphi_5^{-1}(Q) \in \mathbb{H}^2 \subset \mathbb{H}^3$. Let φ_2 & φ_4^{-1} be rotations about vertical lines thru $\varphi_1(P), \varphi_5^{-1}(Q)$, resp., such that $\underline{(D_P(\varphi_2 \circ \varphi_1))(\vec{v}) \text{ \& } (D_Q(\varphi_4^{-1} \circ \varphi_5^{-1}))(\vec{w})} \parallel \mathbb{H}^2 \subset \mathbb{H}^3$.

b/c $\mathrm{Isom}(\mathbb{H}^2)$ is gen'd by nice things

Finally, b/c $\mathbb{H}^2$ is <u>isotropic</u>, $\exists\, \varphi_3 : \mathbb{H}^2 \to \mathbb{H}^2$, extensible to $\mathbb{H}^3 \to \mathbb{H}^3$, s.t. $\underline{(\varphi_3 \circ \varphi_2 \circ \varphi_1)(P) = (\varphi_4^{-1} \circ \varphi_5^{-1})(Q)}$ and $\underline{(D_P(\varphi_3 \circ \varphi_2 \circ \varphi_1))(\vec{v}) = (D_Q(\varphi_4^{-1} \circ \varphi_5^{-1}))(\vec{w})}$.

Now let $\varphi = \varphi_5 \circ \varphi_4 \circ \varphi_3 \circ \varphi_2 \circ \varphi_1$. $\diamond$

Fundamental properties

Thm. The hyperbolic space $(\mathbb{H}^3, d_{hyp})$ is isotropic.

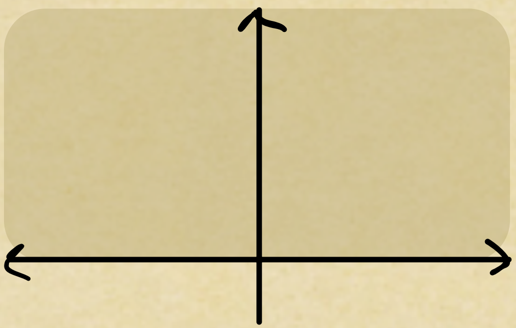
The following two properties are proven exactly as in the 2D case:

Thm. The hyperbolic space $(\mathbb{H}^3, d_{hyp})$ is <u>complete</u>.

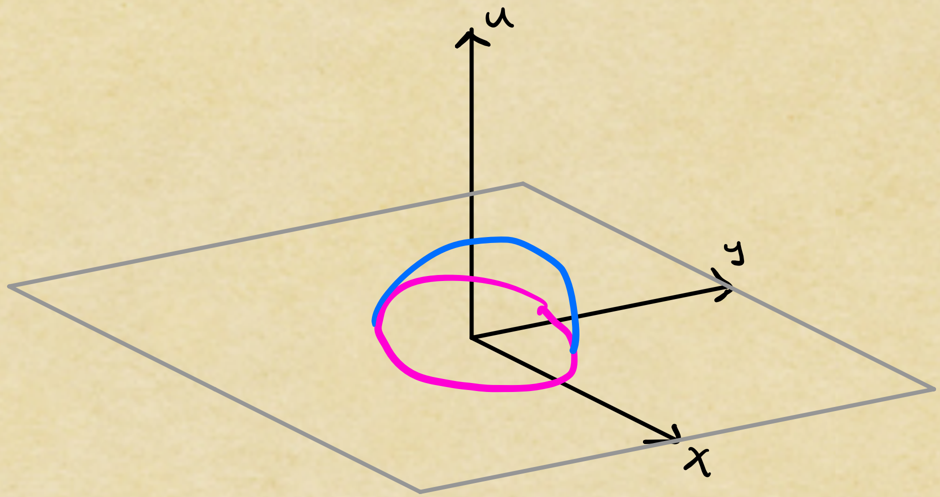
Thm. The shortest curve connecting $P \in \mathbb{H}^3$ to $Q \in \mathbb{H}^3$ is an arc of the <u>vertical circle</u> (possibly a <u>line</u>) passing thru P & Q and centered on the <u>xy-plane</u>.

All isometries of $\mathbb{H}^3$

In order to classify the isometries of $(\mathbb{H}^3, d_{hyp})$, it will be helpful to treat $\hat{\mathbb{C}}$ as the boundary of $\mathbb{H}^3$.
(C.f. $\hat{\mathbb{R}} \simeq \partial\mathbb{H}^2$)



vs.



Lemma Every LFM or ALFM $\varphi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ continuously extends to a map $\hat{\varphi} : \mathbb{H}^3 \cup \hat{\mathbb{C}} \longrightarrow \mathbb{H}^3 \cup \hat{\mathbb{C}}$ whose restriction to $\mathbb{H}^3$ is an isometry of $(\mathbb{H}^3, d_{hyp})$.

(Proof idea.) Every (A)LFM of $\hat{\mathbb{C}}$ is a composition of translations, rotations, homothetics, & inversions across the unit circle. Each extends to $\mathbb{H}^3$. (Think of $u=0$.) $\diamond$

All isometries of $\mathbb{H}^3$

It's not difficult to show that the extension
$\hat{\varphi}: \mathbb{H}^3 \cup \hat{\mathbb{C}} \to \mathbb{H}^3 \cup \hat{\mathbb{C}}$ is uniquely determined by $\varphi: \hat{\mathbb{C}} \to \hat{\mathbb{C}}$.

Key: (A)LFMs on $\hat{\mathbb{C}}$ send spheres in $\mathbb{H}^3$ centered on $\{u=0\}$ to spheres in $\mathbb{H}^3$ centered on $\{u=0\}$ + vertical planes.

Conversely, every isometry of $(\mathbb{H}^3, d_{hyp})$ is obtained in this manner.

Thm. For every isometry ψ of $(\mathbb{H}^3, d_{hyp})$, there is an (A)LFM $\varphi: \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ whose continuous extension $\hat{\varphi}: \mathbb{H}^3 \cup \hat{\mathbb{C}} \to \mathbb{H}^3 \cup \hat{\mathbb{C}}$ which extends ψ.

(Proof.) Let $g_0 \subset \mathbb{H}^3$ denote the complete geodesic from 0 to ∞. Then $\psi(g_0)$ is a complete geodesic from $\psi(0)$ to $\psi(\infty)$, and $\exists$ a LFM $\eta: \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ s.t. $\eta(\psi(0)) = 0$ and $\eta(\psi(\infty)) = \infty$.

in $\partial \mathbb{H}^3 = \hat{\mathbb{C}}$

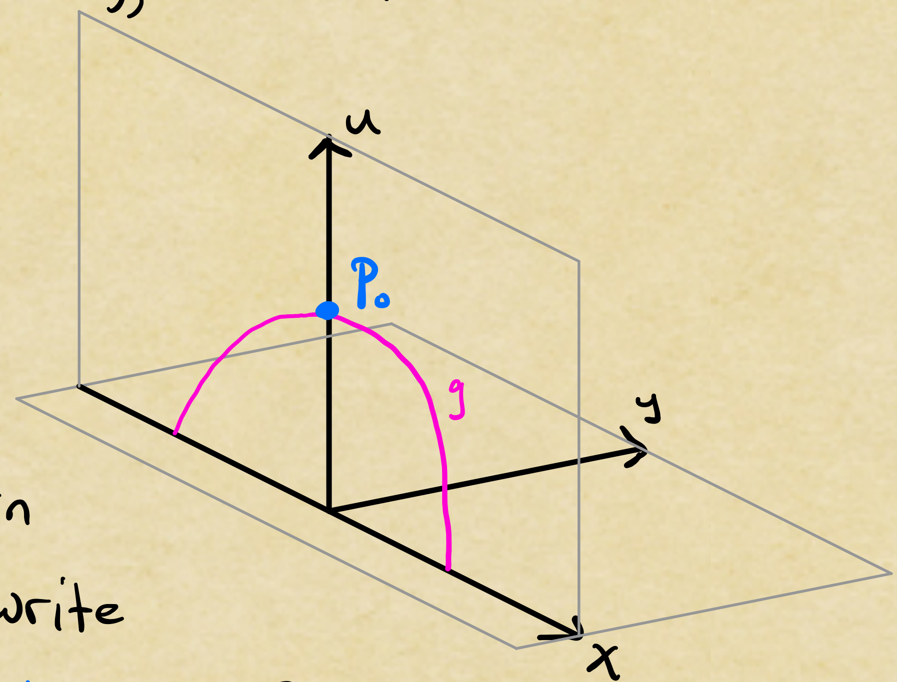
All isometries of $\mathbb{H}^3$

* If $\hat{\eta} \circ \psi$ is an (A)LFM extension, then so is ψ.

Then the extension $\hat{\eta}$ takes $\psi(g_0)$ back to g_0. By replacing* ψ with $\hat{\eta} \circ \psi$ if necessary, we can assume that $\psi(g_0) = g_0$.

We can further compose with a homothety to ensure that

$P \in g_0 \Rightarrow \psi(P) = P$.



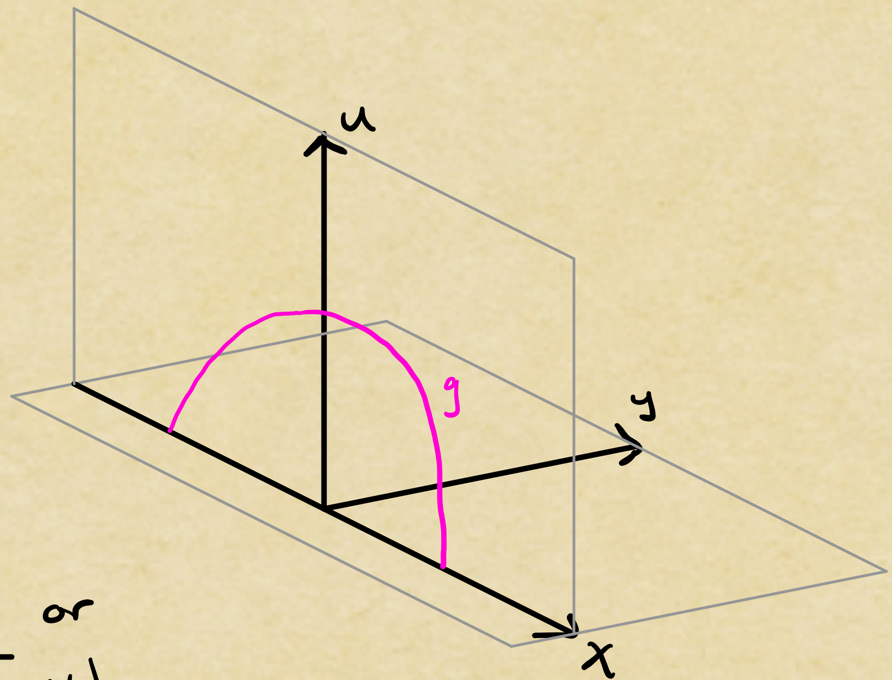
Now let g be a complete geodesic in $\mathbb{H}^2 \subset \mathbb{H}^3$ w/ $g \cap g_0 = P_0$. We can write

$\mathbb{H}^2 = \bigcup \{\text{geodesics which intersect } g_0 \text{ \& } g - \{P_0\}\}$.

So $\psi(\mathbb{H}^2) = \bigcup \{\text{same, but for } \psi(g)\}$ Upshot: $\psi(\mathbb{H}^2)$ is a vertical plane thru g_0. By composing with a rotation about u-axis, we may assume that $\psi(\mathbb{H}^2) = \mathbb{H}^2$.

All isometries of $\mathbb{H}^3$

Since $\Psi : \mathbb{H}^3 \to \mathbb{H}^3$ is an isometry and $\Psi(\mathbb{H}^2) = \mathbb{H}^2$, the restriction $\Psi|_{\mathbb{H}^2}$ is an isometry.



Because $\Psi|_{g_0} \equiv id_{g_0}$, we can assume that $\Psi|_{\mathbb{H}^2} \equiv id_{\mathbb{H}^2}$, possibly after composing with the reflection across g_0.

But then Ψ must either be $id_{\mathbb{H}^3}$ or the reflection across $\mathbb{H}^2$. In either case, $\exists$ an (A) LFM $\varphi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that

$$\hat{\varphi}\,\Big|_{\mathbb{H}^3} = \Psi.$$

$\diamond$

Conclusions

This was a crash course, but the basic takeaway is that $(\mathbb{H}^3, d_{hyp})$ works much like $(\mathbb{H}^2, d_{hyp})$.

Shortest curves are, once again, vertical circle arcs centered on the boundary of the space (incl. vertical lines).

The collection of isometries is again parametrized by (A)LFMs, but now they can have complex coefficients, and we can't plug in points of $\mathbb{H}^3$.