Math 4803

April 3, 2024

LAST TIME

Building Euclidean, spherical, and hyperbolic <u>surfaces</u> via <u>isometric group actions</u>.

TODAY

Relating the group action approach to the <u>edge gluing</u> approach.

Fundamental domains

Unless stated otherwise, assume that (X,d) is $(\mathbb{R}^2, d_{euc})$, $(\mathbb{H}^2, d_{hyp})$, or (S^2, d_{sph}).

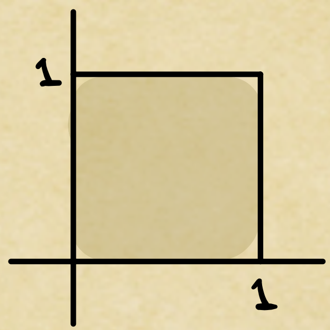
Given a group action $\Gamma \curvearrowright (X,d)$, a fundamental domain for the action is a connected polygon $\Delta \subset X$ s.t.

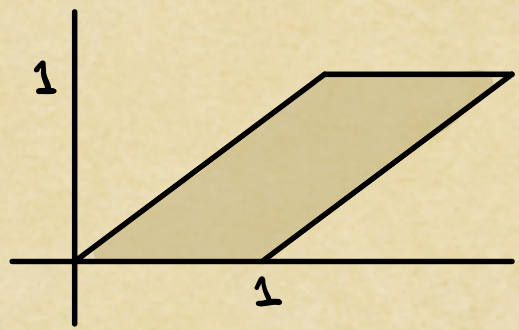
$$\{\gamma(\Delta) \mid \gamma \in \Gamma\}$$

forms a tessellation* of (X,d).

__Ex__. Consider the action $\mathbb{Z}^2 \curvearrowright (\mathbb{R}^2, d_{euc})$

$$(m,n).(x,y) = (x+m,\ y+n).$$

Both 

and 

are fundamental domains for this action.
Any others?

Fundamental domains

Because tessellations are locally finite, actions which admit fundamental domains are discontinuous.

Prop. Any isometric action on $(\mathbb{R}^2, d_{euc})$, $(\mathbb{H}^2, d_{hyp})$, or (S^2, d_{sph}) which admits a fundamental domain is discontinuous.

(Proof.) Denote the action by $\Gamma \curvearrowright (X, d)$ and consider $P \in X$. We NTS that $\exists\, \varepsilon > 0$ s.t. $\{\gamma \in \Gamma \mid \gamma(P) \in B_d(P, \varepsilon)\}$ is finite.

Because $\{\gamma(\Delta) \mid \gamma \in \Gamma\}$ is a tessellation, local finiteness ensures that $\{\gamma \in \Gamma \mid \gamma(\Delta) \cap B_d(P, \varepsilon) \neq \emptyset\}$ is finite, for some $\varepsilon > 0$. Write this set as $\{\gamma_1, \gamma_2, \dots, \gamma_n\}$ and $\oint$ WLOG that $P \in \gamma_1(\Delta)$. Now for any $\gamma \in \Gamma$,

$$\gamma(P) \in B_d(P, \varepsilon) \Rightarrow (\gamma \circ \gamma_1)(\Delta) \cap B_d(P, \varepsilon) \neq \emptyset \Rightarrow \exists\, j \text{ s.t. } \gamma \circ \gamma_1 = \gamma_j$$
$$\Rightarrow \gamma = \gamma_j \circ \gamma_1^{-1}.$$

So there are finitely many $\gamma \in \Gamma$ s.t. $\gamma(P) \in B_d(P, \varepsilon)$. $\diamond$

Fundamental domains

Because tessellations are locally finite, actions which admit fundamental domains are discontinuous.

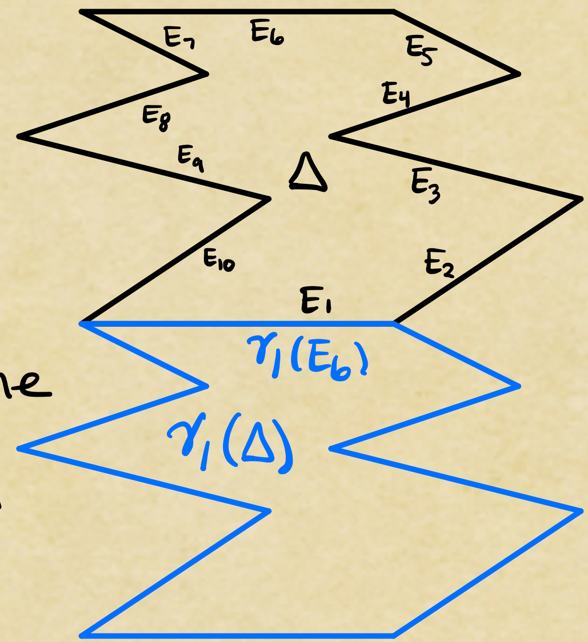
Prop. Any isometric action on $(\mathbb{R}^2, d_{euc})$, $(\mathbb{H}^2, d_{hyp})$, or (S^2, d_{sph}) which admits a fundamental domain is discontinuous.

Cor. If (X,d) is $(\mathbb{R}^2, d_{euc})$, $(\mathbb{H}^2, d_{hyp})$, or (S^2, d_{sph}) and $\Gamma \curvearrowright (X,d)$ is an isometric group action which admits a fundamental domain, then $(X/\Gamma, \bar{d})$ is a metric space.

Q. We've previously built metric spaces as quotients of polygons. If $\Gamma \curvearrowright (X,d)$ has fundamental domain $\Delta \subset X$, is there a way to build $(X/\Gamma, \bar{d})$ from Δ?

Fundamental domains

§ $\Delta \subset X$ is a fundamental domain for $\Gamma \curvearrowright (X,d)$ with edges $E_1, \dots, E_n$.

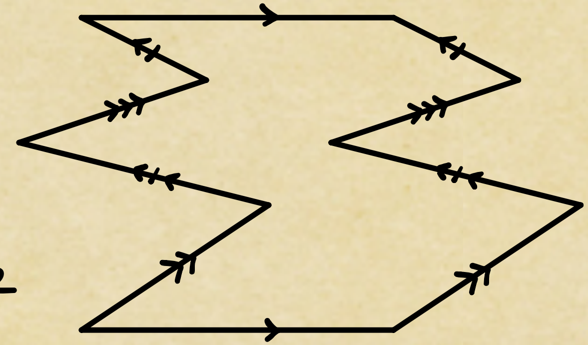


Each E_i separates Δ from $\underline{\gamma_i(\Delta)}$, for some $\gamma_i \in \Gamma$. This means there's some edge $E_{j_i} \subset \Delta$ s.t. $\underline{E_i = \gamma_i(E_{j_i})}$. Then we can let $\varphi_i = \gamma_i^{-1} \in \Gamma$ and notice that $\{\varphi_i \mid 1 \le i \le n\}$ gives an <u>edge gluing</u> for Δ.

b/c we want $\varphi_i(E_i) = E_{j_i}$

<u>Thm.</u> Let (X,d) be $(\mathbb{R}^2, d_{euc})$, $(\mathbb{H}^2, d_{hyp})$, or (S^2, d_{sph}), and let $\Gamma \curvearrowright (X,d)$ be an isometric action with fundamental domain Δ. Then Γ is generated by $\{\varphi_i \mid 1 \le i \le n\}$, where Δ has edges $E_1, \dots, E_n$ and φ_i is built as above.

Fundamental domains



Now that we have an edge gluing on Δ, we can build a quotient metric space $(\hat{\Delta}, \hat{d}_\Delta)$. This construction relied on the same data we used to build $(X/\Gamma, \bar{d})$, so the following result might not surprise you.

Thm. Let (X,d) be $(\mathbb{R}^2, d_{euc})$, $(\mathbb{H}^2, d_{hyp})$, or (S^2, d_{sph}), and let $\Gamma \curvearrowright (X,d)$ be an isometric action with fundamental domain Δ. Then

$$(\hat{\Delta}, \hat{d}_\Delta) \quad \text{and} \quad (X/\Gamma, \bar{d})$$

are isometric to one another.

Dirichlet domains

We know that isometric actions which admit fundamental domains are discontinuous. Is the converse true?

Sort of... we must allow Δ to have infinitely many edges. But Δ is still required to be locally finite, in that, for every $P \in \Delta$, $\exists\, \varepsilon > 0$ s.t.

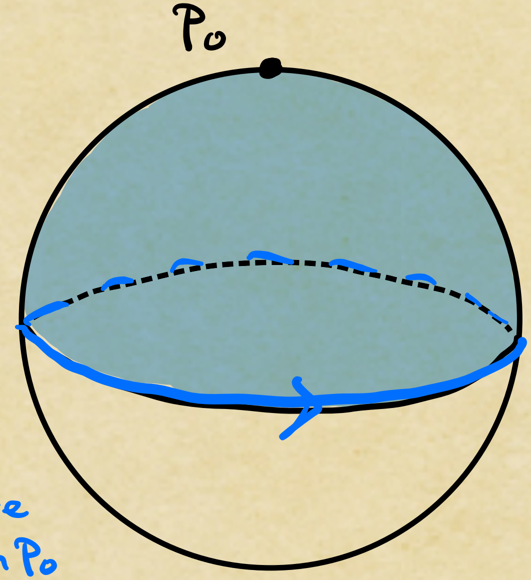
$$B_d(P,\varepsilon) \cap \{\text{edges}\} \quad \& \quad B_d(P,\varepsilon) \cap \{\text{vertices}\}$$

are finite sets.

Given a discontinuous action $\Gamma \curvearrowright (X,d)$ and a point $P_0 \in X$, we define the Dirichlet domain

$$\Delta_\Gamma(P_0) = \{ P \in X \mid d(P,P_0) \leq d(P,\gamma(P_0)),\ \forall \gamma \in \Gamma \}.$$

Dirichlet domains



Ex. Consider the action $\mathbb{Z}/2\mathbb{Z} \curvearrowright (S^2, d_{sph})$ defined by $e.P = P$ & $a.P = -P$, where $\mathbb{Z}/2\mathbb{Z} = \{e, a\}$. For any $P_0 \in (S^2, d_{sph})$,

$$\Delta_\Gamma(P_0) = \{P \in S^2 \mid d(P, P_0) \leq d(P, -P_0)\} = $$ closed hemisphere centered on P_0

Thm Let (X,d) be $(\mathbb{R}^2, d_{euc})$, $(\mathbb{H}^2, d_{hyp})$, or (S^2, d_{sph}), and let $\Gamma \curvearrowright (X,d)$ be a discontinuous isometric action. Then for every $P_0 \in X$, the Dirichlet domain $\Delta_\Gamma(P_0)$ is a locally finite polygon and

$$\{\gamma(\Delta_\Gamma(P_0)) \mid \gamma \in \Gamma\}$$

is a tessellation (X,d). Moreover, if $\Gamma_{P_0} = \{id_X\}$, then $\Delta_\Gamma(P_0)$ is a fundamental domain for $\Gamma \curvearrowright (X,d)$.

Conclusions from chapters 4-7

We like to build quotients of $(\mathbb{R}^2, d_{euc})$, $(\mathbb{H}^2, d_{hyp})$, and (S^2, d_{sph}), as well as tessellations of these spaces. Quotients which are locally isometric to the original are especially nice.

Two methods for building these:

(1) Edge gluings of polygons. Whether building surfaces or tessellations, there's an angle condition to check. For tessellations, we also have a completeness criterion.

(2) Discontinuous group actions. We can build what we want from these when they have small stabilizer subgroups.

Fundamental domains represent an overlap in these two approaches, but the symmetric difference is nonempty.

Next

3D!