Math 4803 April 22, 2024

LAST TIME

- Uniqueness for geometrization of Knot complements.
- Geometrization for surfaces. } Uniformization

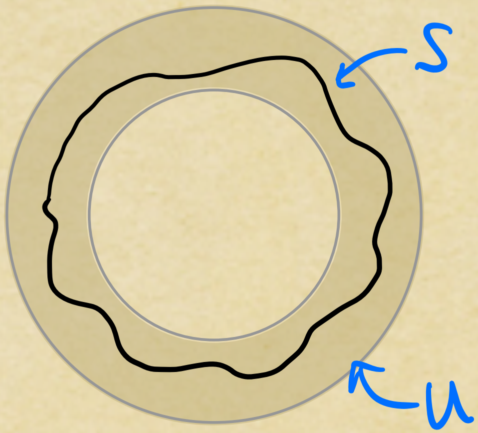
TODAY (LAST TIME)

- Geometrization for 3-manifolds of finite topological type.

Essential surfaces in 3-manifolds

Let (X,d) be a 3-manifold. Our decomposition of (X,d) will involve cutting along essential surfaces — roughly, surfaces where cutting isn't boring.

A two-sided surface in X is a subset $S \subset X$ s.t.



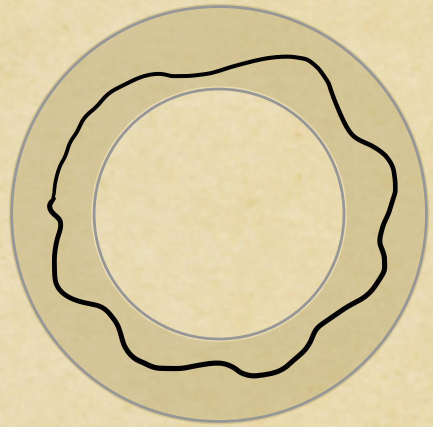
$\exists$
- a 2-manifold Y;
- a subset $U \subset X$;
- a homeomorphism $\varphi : Y \times (-\varepsilon, \varepsilon) \longrightarrow U$,

for some $\varepsilon > 0$, such that $\varphi(Y \times \{0\}) = S$

But (X,d) contains lots of boring two-sided surfaces: consider boundaries of small balls.

Essential surfaces in 3-manifolds

To cut along a surface, we want it to be two-sided, but not boring.

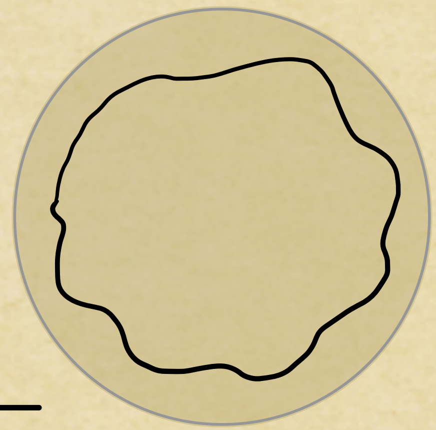


Let (X,d) be a metric space which is a 3-manifold. A subset $S \subset X$ is an **essential sphere** if:

(1) S is homeomorphic to (S^2, d_{sph});

(2) S is two-sided;

(3) S is not contained in a subset $B \subset X$ homeomorphic to $(\mathbb{R}^3, d_{euc})$.



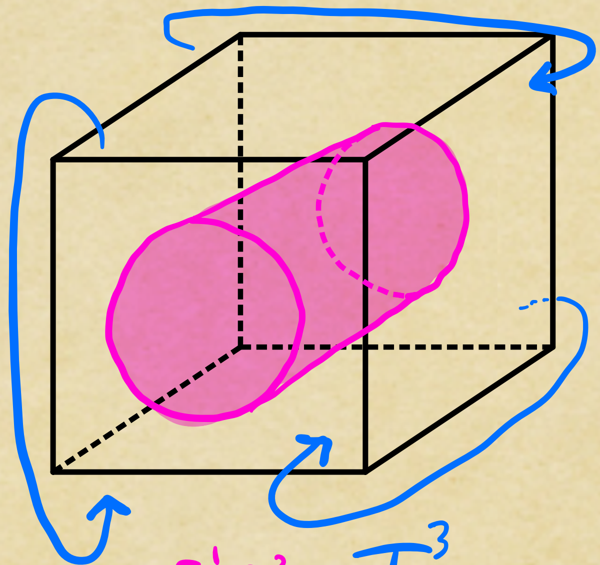
A subset $P \subset X$ is an **essential projective plane** if

(1) P is homeomorphic to $(\mathbb{RP}^2, \bar{d}_{sph})$;

(2) P is two-sided.

Essential surfaces in 3-manifolds

Defining "not boring" for tori is a bit more involved, because tori can bound regions not contained in balls.



A subset $T \subset X$ is called an **essential torus** if:

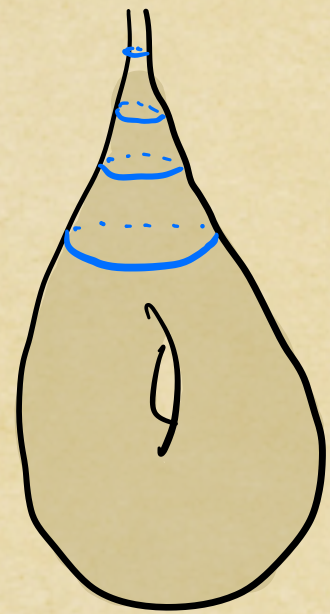
(1) T is homeomorphic to $(T^2, \bar{d}_{euc})$;

(2) T is two-sided;

(3) T is not contained in a subset $B \subset X$ which is homeomorphic to $(\mathbb{R}^3, d_{euc})$; (?)

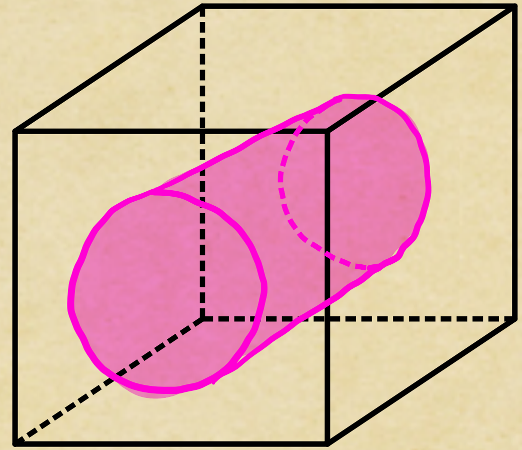
(4) T does not bound a solid torus in X;

(5) T does not bound a subset of X homeomorphic to $T^2 \times [0, \infty)$.



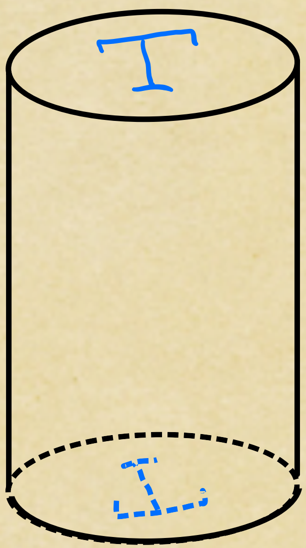
Essential surfaces in 3-manifolds

Let's be precise about bounding requirements. We'll say that $T \subset X$ **bounds a solid torus $V \subset X$** if



there's a homeomorphism

$\varphi : (S^1 \times D^2, d_{euc}) \to (V, d|_V)$ s.t. $\varphi(S^1 \times S^1) = T$.



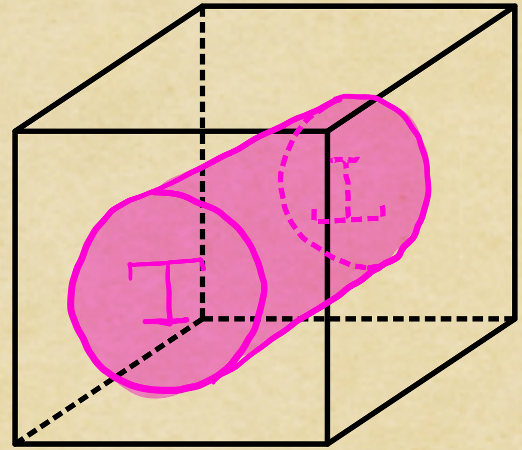
A more complicated sort of solid is the twisted solid torus $S^1 \tilde{\times} D^2$, obtained as the quotient $([0,1] \times D^2)/\sim$, where $(1, P) \sim (0, \overline{P})$, for every $P \in D^2$.

We say that $K \subset X$ bounds a twisted solid torus $V \subset X$ if there's a homeomorphism

$\varphi : (S^1 \tilde{\times} D^2, d_{euc}) \to (V, d|_V)$ s.t. $\varphi(S^1 \tilde{\times} S^1) = K$.

Essential surfaces in 3-manifolds

A subset $K \subset X$ is called an essential Klein bottle if:



(1) K is homeomorphic to $(K^2, \bar{d}_{euc})$;

(2) K is two-sided;

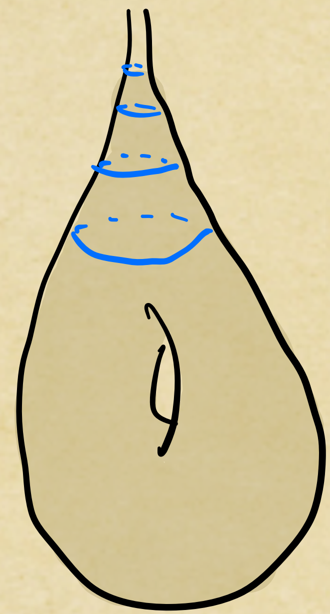
(3) K does not bound a twisted solid torus in X;

(4) K is not the boundary of an infinite collar

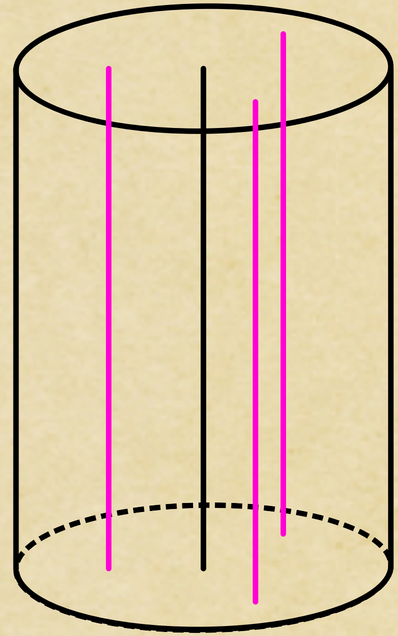
i.e., $\nexists$ a subset $W \subset X$ with a homeomorphism $\varphi : K^2 \times [0,\infty) \to W$ such that $K = \varphi(K^2 \times \{0\})$ and s.t.

$$\lim_{t\to\infty} \inf\{d(P_0, P) \mid P \in \varphi(K^2 \times \{t\})\} = \infty,$$

for some $P_0 \in W$.



Seifert manifolds



Our statement of the geometrization theorem will make exceptions for 3-mflds containing essential surfaces.

But these are not the only exceptions. There are also Seifert fibered spaces.

Consider partitioning the solid cylinder $[0,1]\times D^2$ into lines $[0,1]\times\{x\}$ and gluing via $(1,P)\sim(0,R_{p/q\cdot 2\pi}(P))$. The result is a solid torus partitioned by circles.

Most of these intersect the disk $\{0\}\times D^2$ q times, but $[0,1]\times\{0\}$ is an exceptional fiber.

The twisted solid torus is another fibered torus. Its regular fibers have length 2.

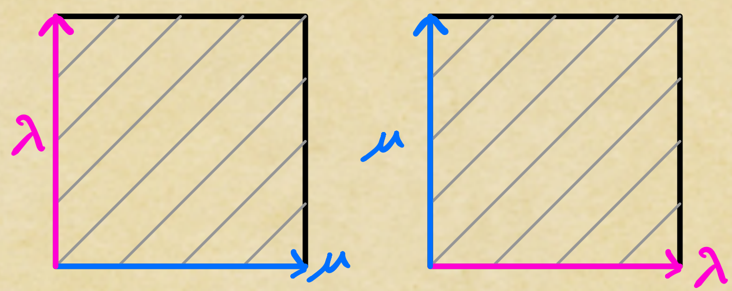
Seifert manifolds

Def'n: A Seifert fibration of a 3-manifold X is a partition of X into circles — called fibers — such that, for every circle K, there exist

- a subset $U \subset X$ containing K;
- a rational number $p/q \in \mathbb{Q}$;
- a homeomorphism φ from U to either the solid torus $S^1 \times D^2$ or the twisted solid torus $S^1 \tilde{\times} D^2$;

s.t. (1) U is a union of fibers;
(2) φ sends K to the core of the image (twisted) solid torus;
(3) φ sends the partition of X to the p/q-partition of the solid torus or the standard partition of the twisted solid torus.

Ex The 3-sphere can be split into two solid tori, each with the $1/1$-fibration.



The general geometrization theorem

One last bit of vocab: we call (X,d) a manifold-with-boundary if it's locally homeomorphic to $\mathbb{R}^n_+ = \mathbb{R}^{n-1} \times [0,\infty)$.
The boundary ∂X corresponds to $\mathbb{R}^{n-1} \times \{0\}$

A manifold has finite topological type if it's homeomorphic to $X - \partial X$ for some compact manifold-with-boundary X.

Thm (Thurston, Hamilton, Perelman, etc)

Let (X,d) be a connected 3-dimensional manifold with finite topological type. Then at least one of these holds:

(1) X contains an essential S^2, $\mathbb{RP}^2$, T^2 or K^2;

(2) X admits a Seifert fibration;

(3) X admits a complete, hyperbolic metric d' s.t. $id: (X,d) \longrightarrow (X,d')$ is a homeomorphism.

The end