Math 4803 April 17, 2024

LAST TIME

For every knot $K \subset S^3$, exactly one of the following is true:

- K is a torus knot;
- K is a nontrivial satellite of a nontrivial knot;
- $S_K := S^3 - K$ admits a very nice hyperbolic metric d_K.

TODAY

① Getting knot invariant out of this.

② A general result of this sort in 2D.

③ Towards the 3D result.

Geometrization for Knot Complements

Thm (Thurston, 1974)

For any knot $K \subset S^3$, exactly one of the following is true:

(1) K is a torus knot $T(p,q)$ with $q \geq 2$;

(2) K is a nontrivial satellite of a nontrivial knot;

(3) there is a metric d_K on $S^3 - K$ such that

(a) the map
$$(S^3-K, d_{sph}) \longrightarrow (S^3-K, d_K)$$
$$p \longmapsto p$$
is a homeomorphism;

(b) (S^3-K, d_K) is complete;

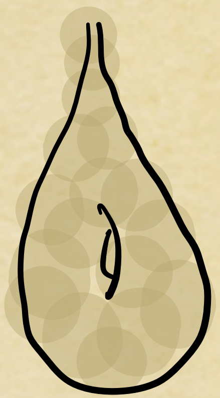
(c) (S^3-K, d_K) is locally isometric to $(\mathbb{H}^3, d_{hy})$.

We call K a <u>hyperbolic knot</u> in case (3).

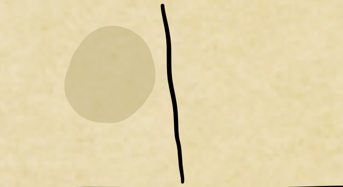
Q: Is (S_K, d_K) an invariant of K?

Mostow's Rigidity Theorem

Def: Let (X,d) be locally isometric to $(\mathbb{H}^n, d_{hyp})$.
We say that (X,d) has <u>finite volume</u> if we can pick balls $B_d(P_n, r_n) \subset X$ & $B_{d_{hyp}}(Q_n, r_n) \subset \mathbb{H}^n$ such that (1) for all $n \in \mathbb{N}$,



$$B_d(P_n, r_n) \overset{\text{isometry}}{\cong} B_{d_{hyp}}(Q_n, r_n);$$



(2) $X = \bigcup_{n \in \mathbb{N}} B_d(P_n, r_n)$;

(3) the series

$\sum_{n \in \mathbb{N}} \mathrm{vol}(B_{d_{hyp}}(Q_n, r_n))$ converges.

If (X,d) has finite volume, we can carefully define its volume, but won't.

Fact: If $K \neq U$, then (S_K, d_K) has finite volume.

Mostow's Rigidity Theorem

*In fact, a weaker hypothesis suffices.

Thm (Mostow) Fix $n \geq 3$ and let (X, d) and (X', d') be complete metric spaces which are locally isometric to $(\mathbb{H}^n, d_{hyp})$ and which have finite volume. If (X, d) and (X', d') are homeomorphic*, then (X, d) and (X', d') are isometric.

Remark: The most common proof uses plenty of algebraic topology, so we won't touch it.

Cor. If hyperbolic knots K and K' are isomorphic, then their hyperbolic complements (S_K, d_K) and $(S_{K'}, d_{K'})$ are isometric.

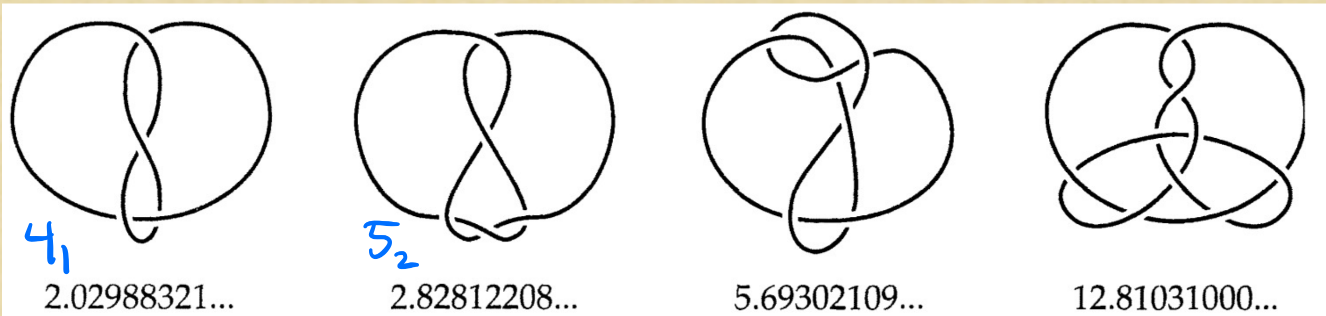
(Proof.) By the definition of isomorphism, K & K' have homeomorphic complements; the other hypotheses are given by Thurston's metrics d_K & $d_{K'}$. ◇

Mostow's Rigidity Theorem

The actionable consequence is that if hyperbolic knots $K, K' \subset S^3$ have complements which are not isometric, then K & K' are not isomorphic.

(and thus not isotopic).

Namely, if (S_K, d_K) & $(S_{K'}, d_{K'})$ have distinct volumes, then K and K' are distinct knots.



(Image via Adams)

If $K \subset S^3$ is a hyperbolic knot, we refer to $\mathrm{vol}(S_K, d_K)$ as the volume of K.

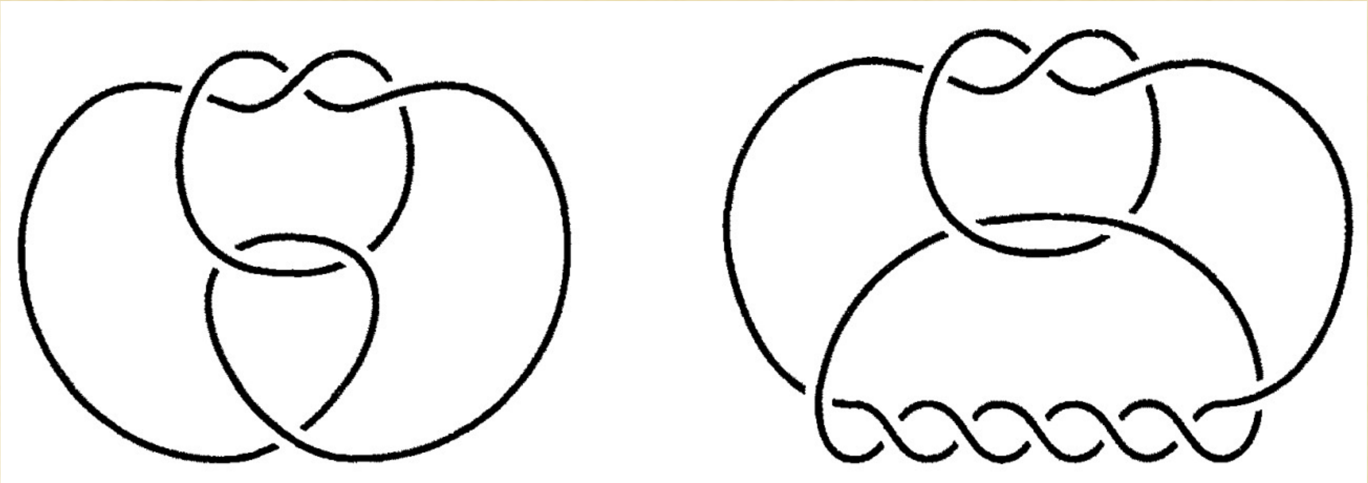
SnapPy ← Built on SnapPea

SnapPy can often compute these for you.

Mostow's Rigidity Theorem

Of course, volume can't tell us everything.

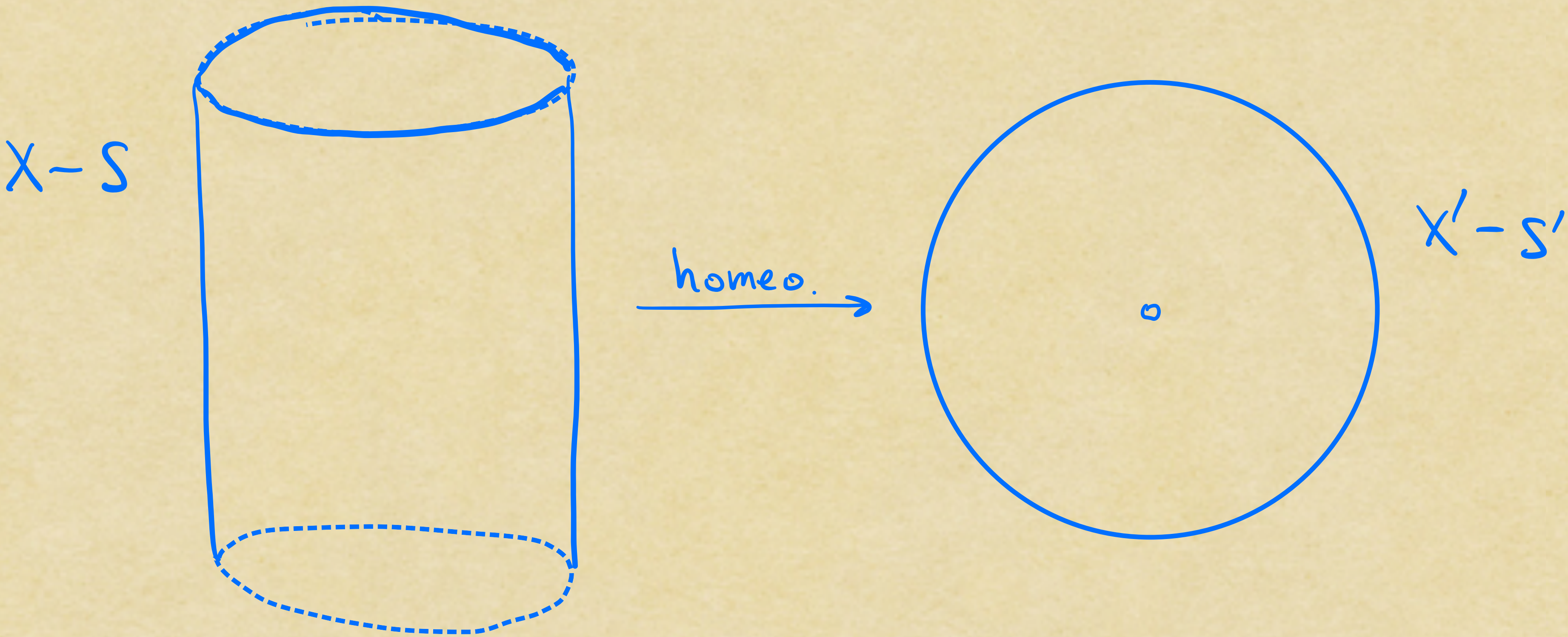
The following knots have the same volume, but can be distinguished by other means:

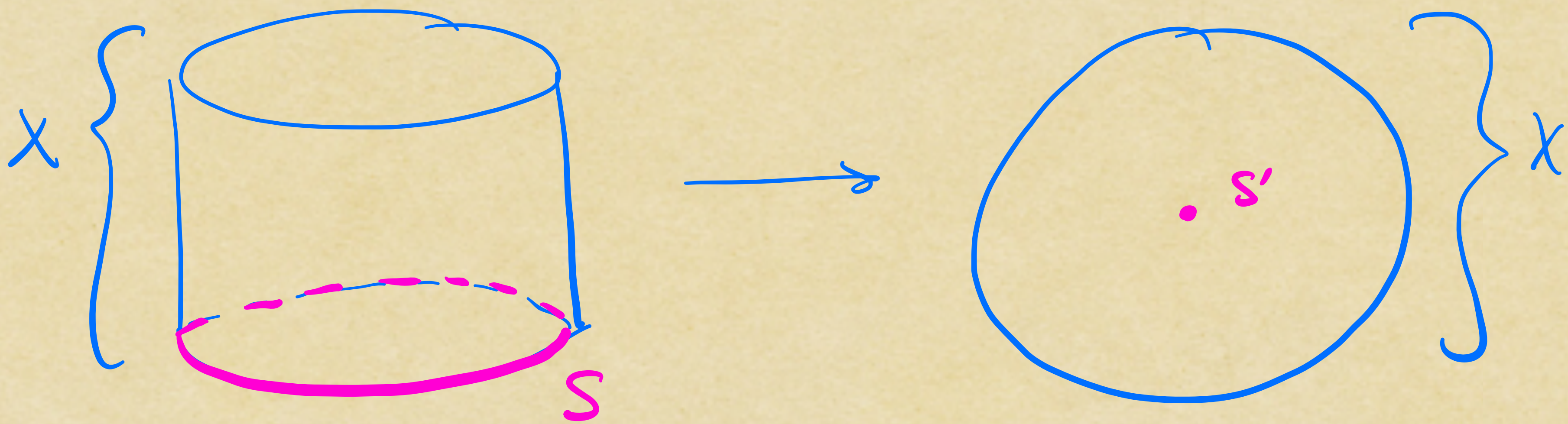


5_2 2.828122 08... (Image via Adams.)

Fact: S_K & $S_{K'}$ are not even homeomorphic!

See the Gordon-Luecke Theorem.


X − S
homeo.
X′ − S′


X
S
S′
X′

Towards geometrization for manifolds

We'll state two more geometrization results, both for spaces with the local topological appearance of Euclidean space.

Euclidean n-space is the metric space $(\mathbb{R}^n, d_{euc})$, where

$$d_{euc}(\vec{P}, \vec{Q}) := \sqrt{\sum_{i=1}^{n} (P_i - Q_i)^2}.$$

* NOT the usual definition

A metric space (X,d) is an n-dimensional manifold* if: for every $P \in X$, there is a ball $B_d(P,\varepsilon)$ and a homeomorphism $\varphi : B_d(P,\varepsilon) \to B_{d_{euc}}(Q,\varepsilon') \subset \mathbb{R}^n$.

We'll call a manifold (X,d) connected if every pair of points $P, Q \in X$ admits a continuous curve $\gamma : [a,b] \to X$ with $\gamma(a) = P$ and $\gamma(b) = Q$.

Geometrization for 2-manifolds

Thm (Geometrization for surfaces) Let (X,d) be a metric space which is a connected 2-dimensional manifold. Then there is a metric d' on X such that

(1) the map $(X,d) \longrightarrow (X,d')$

$$p \longmapsto p$$

is a homeomorphism;

(2) (X,d') is complete;

(3) (X,d') is locally isometric to $(\mathbb{R}^2, d_{euc})$, (S^2, d_{sph}), or $(\mathbb{H}^2, d_{hyp})$.

Rmk: Unlike in the knot complement case, d' is not necessarily unique. (No Mostow rigidity for $n=2$.)

Ex: 

Geometrization for 2-manifolds

In fact, we can say even more: Let (X,d') be a metric space satisfying the conclusions of the geometrization theorem for surfaces.

Then
- if (X,d') is locally isometric to (S^2, d_{sph}), it's homeomorphic to S² or $\mathbb{R}P^2$; projective plane
- if (X,d') is locally isometric to $(\mathbb{R}^2, d_{euc})$, it's homeomorphic to $\mathbb{R}^2$, the cylinder, the torus, the Möbius strip, or the Klein bottle;
- if (X,d') is locally isometric to $(\mathbb{H}^2, d_{hyp})$, it's NOT homeomorphic to S^2, $\mathbb{R}P^2$, the torus, or the Klein bottle.

Lesson: Hyperbolic geometry is everywhere!!

Geometrization for 3-manifolds

The most natural generalization to 3D is false:

Dashed hope: ~~If (X, d) is a connected 3-dimensional mfld, it admits a topologically equivalent, complete metric d' such that (X, d') is locally isometric to a model geometry.~~

The next best thing is to decompose (X, d) into pieces which admit these nice metrics.

Our next task is to determine the meaning of the word "decompose."