Math 4803

April 15, 2024

LAST TIME

Basic definitions and examples
for Knot theory.

TODAY

Hyperbolic metrics on Knot complements.

(Warning: Slides written while airborne.)

- CIOS
- Term paper draft due tonight!
- OH @ 9:30am ET on Thursday
- Art Show Apr 24 @ 12pm

Geometrization for Knot Complements

Thm (Thurston, 1974)

For any knot $K \subset S^3$, exactly one of the following is true:

(1) K is a torus knot $T(p,q)$ with $q \geq 2$;

(2) K is a nontrivial satellite of a nontrivial knot;

(3) there is a metric d_K on $S^3 - K$ such that

(a) the map
$$\begin{aligned}(S^3-K, d_{sph}) &\to (S^3-K, d_K)\\ p &\mapsto p\end{aligned}$$
is a homeomorphism;

"d_K induces the same topology as d_{sph}"

(b) (S^3-K, d_K) is complete;

(c) (S^3-K, d_K) is locally isometric to $(\mathbb{H}^3, d_{hyp})$.

Thematically appropriate interpretation: we can build lots of complete metric spaces locally isometric to $\mathbb{H}^3$.

Better interpretation soon.

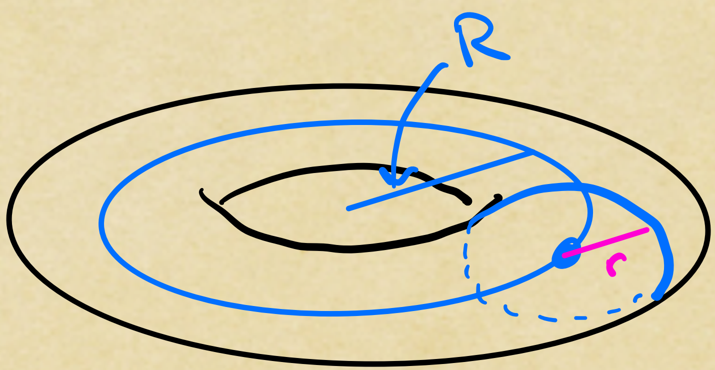
Torus Knots

We'll define the torus knot $T(p,q)$ for any coprime $p, q \in \mathbb{Z}$ not equal to zero.

Given $R > r > 0$, the map $f: \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$f(\theta,\phi) = ((R + r\cos\phi)\cos\theta, (R + r\cos\phi)\sin\theta, r\sin\phi)$$

has image a torus with central radius R and inner radius $r < R$.



The knot $T(p,q)$ is parametrized by

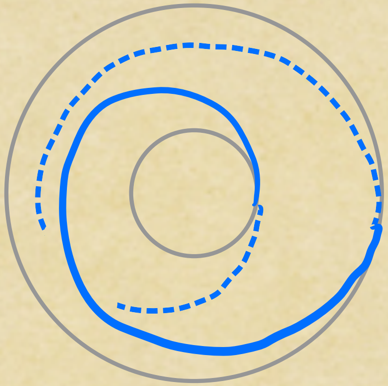
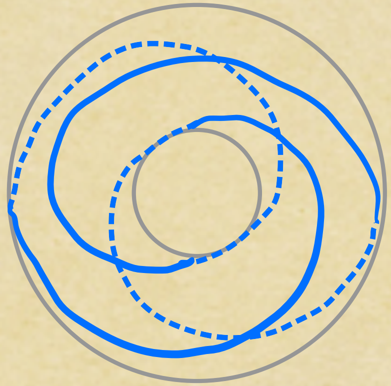
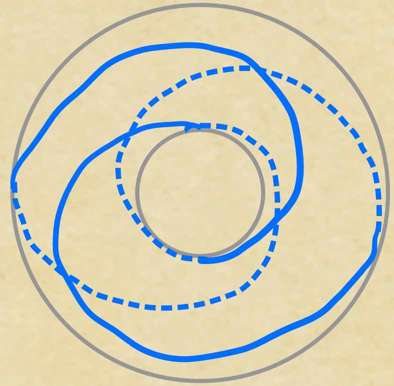
$$\gamma(t) := f(qt, pt).$$

Note that the choice of $R > r > 0$ doesn't matter.

Why do we require $\underset{\text{coprime}}{(p,q) = 1}$?

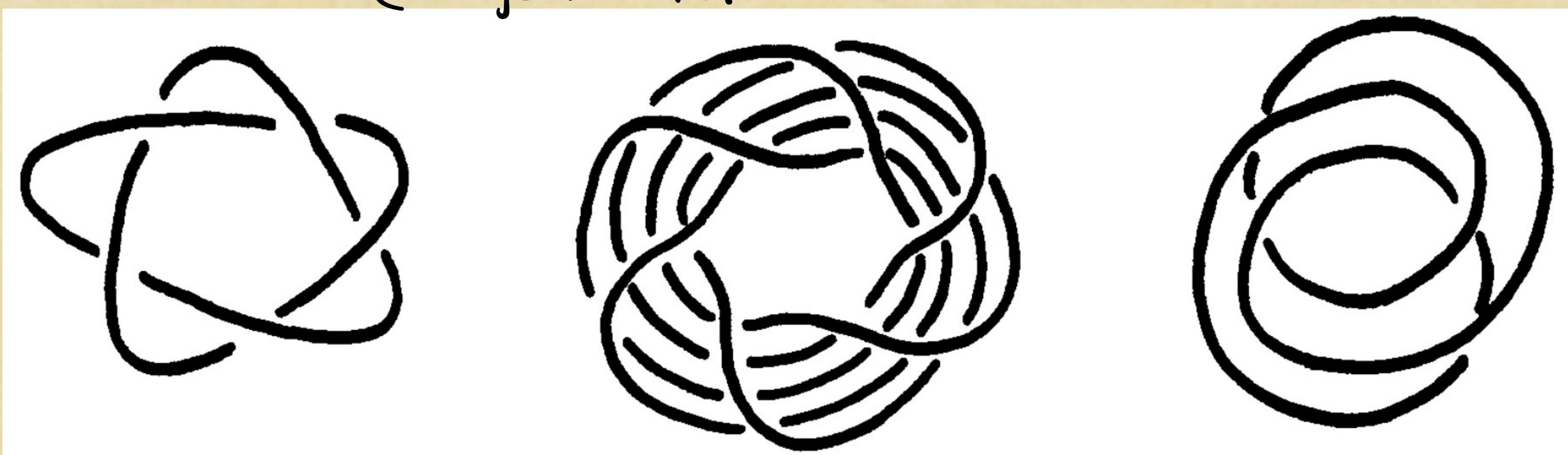
Torus Knots

Ex.

$T(2,1)$ = unknot $T(3,2)$ = LHT $T(-3,2)$ = RHT

(Image from Rolfsen)



$T(2,5)$ $T(5,6)$ $T(3,2)$

Torus Knots

Some facts:

- $T(p, \pm 1)$ and $T(\pm 1, q)$ are isotopic to unknot
- $T(q,p)$ and $T(p,q)$ are isotopic. Exercise: Prove this
- $T(-p,q)$ and $T(p,q)$ are isomorphic.
- If $|p|, |p'|, |q|, |q'| > 1$, then $T(p,q)$ and $T(p',q')$ are isomorphic iff $\left|p'/q'\right| = \left|p/q\right|$ OR $\left|q/p\right|$ and isotopic iff $p'/q' = p/q$ OR q/p.

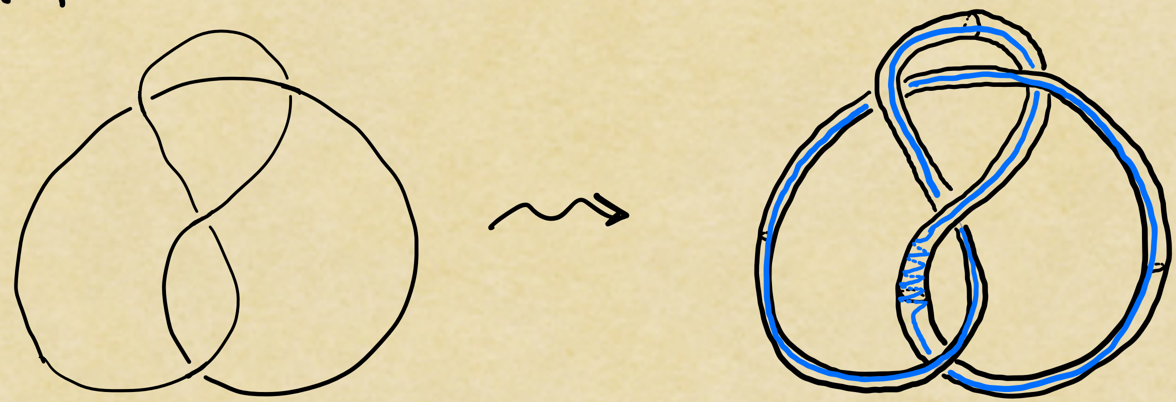
(Proven by computing $\pi_1(S^3 - T(p,q)) = \langle a, b \mid a^p = b^q \rangle$.)

Upshot: Lots of nontrivial knots!

Satellite Knots

Here's a different way of using a torus to build new knots.

Any knot $K \subset S^3$ admits a tubular neighborhood which is a solid torus:

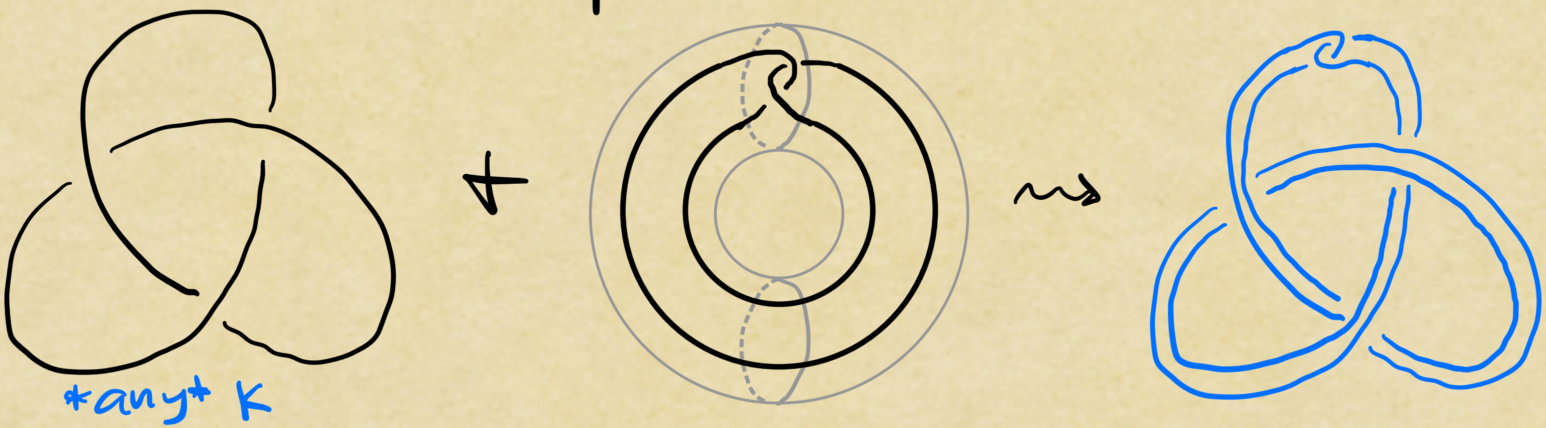


We can trace out the pattern of any torus knot on the boundary of this tubular nbhd to get a (potentially) new knot.

In fact, any pattern in the standard solid torus can be used to obtain a knot K' close to K.

Satellite Knots

Ex. A famous example is the Whitehead double of K:



Def'n. A nontrivial pattern in the standard solid torus $V \subset S^3$ is a knot (up to isotopy) $L \subset V$ which

- not the core/central curve of the torus;
- intersects every meridional disk.

For any knot $K \subset S^3$, let $\Phi : V \longrightarrow N(K)$ be a diffeomorphism from V onto a tubular nbhd of K. Then the satellite of K with pattern L is $K' := \Phi(L)$.

Geometrization for Knot Complements

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(a) the map
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$$p \longmapsto p$$
is a homeomorphism;

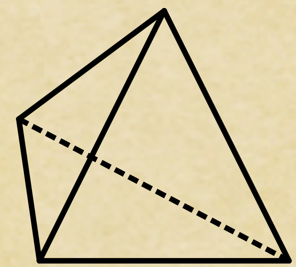
(b) $(S^3 - K, d_K)$ is complete;

(c) $(S^3 - K, d_K)$ is locally isometric to $(\mathbb{H}^3, d_{hyp})$.

Fact: I don't really know anything about the proof.

Geometrization for Knot Complements

The basic idea for constructing d_K follows this recipe:



① Cut $S^3 - K$ into tetrahedra.
(topologically — no geometry yet)

② Realize each tetrahedron in $(\mathbb{H}^3, d_{hyp})$ so that its edges are geodesics and its faces are spheres centered on the xy-axis.

③ Deform each of these hyperbolic tetrahedra until each gluing from ① can be accomplished geometrically.

SnapPea

If n = # of tetrahedra, Step ③ requires solving n polynomial equations in n complex variables :(

Geometrization for Knot Complements

Once we have Thurston's geometrization theorem, we can hope to study (most) knots by studying the geometry of $(S^3_K, d_K) := (S^3 - K, d_K)$.

For this to be an invariant of K, however, we need to be sure that d_K doesn't depend on any of the choices involved (including the representative of the knot K!).

This will follow from Mostow's rigidity theorem, which we'll state next.