# Math 4803

# April 10, 2024

## LAST TIME

The geometry of <u>hyperbolic</u> 3-space.

## TODAY

Some basic definitions and examples for <u>knot theory</u> in the 3-sphere.

$$\text{3-sphere} = \mathbb{R}^3 \cup \{\infty\}$$

# Knots in $S^3$: definitions & examples

We'll write $S^3$ for the set $\underline{\mathbb{R}^3 \cup \{\infty\}}$ with its "obvious" topology.
We don't need a metric right now, but there are lots we could build which are compatible with this topology.

A Knot in $S^3$ is a $\underline{\text{regular}}$, $\underline{\text{simple}}$, $\underline{\text{closed}}$ curve $K \subset \mathbb{R}^3 \subset S^3$.

regular: $\exists$ a differentiable map $\gamma : \mathbb{R} \to \mathbb{R}^3$ s.t.
$\gamma(t) \in K$ & $\gamma'(t) \neq \vec{0}$, $\forall\, t \in \mathbb{R}$.
point: tangent lines make sense

closed: for the $\gamma$ above, $\exists\, T \in \mathbb{R}$ s.t.
$$\gamma(t+T) = \gamma(t), \quad \forall\, t \in \mathbb{R}.$$

Simple: for $t, t' \in [0, T)$, $\gamma(t) = \gamma(t')$
iff
$t = t'$

# Knots in $S^3$: definitions & examples

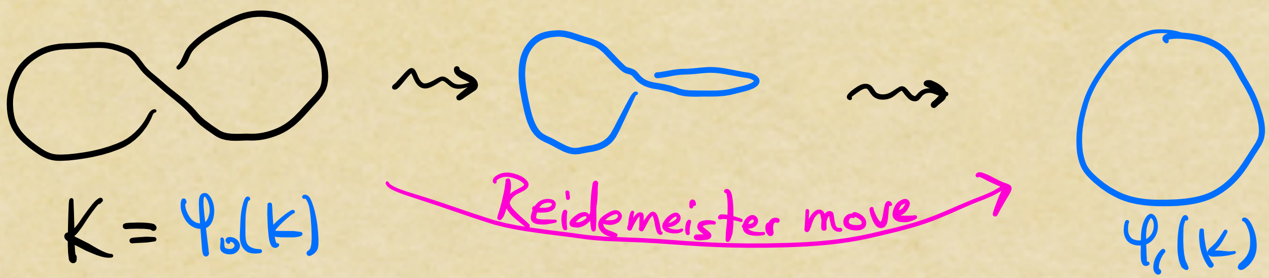
* whatever that means

§ $\varphi_t : \mathbb{R}^3 \to \mathbb{R}^3$, $t \in [0,1]$, is a family of homeomorphisms which depends continuously* on the parameter $t$, with $\varphi_0 \equiv id_{\mathbb{R}^3}$.

$\varphi_t \circ \gamma$ is a parametrization

Then for any knot $K \subset S^3$, $\varphi_0(K) = K$ and each $\varphi_t(K)$ is just a "continuously wiggled" version of $K$.

Ex.

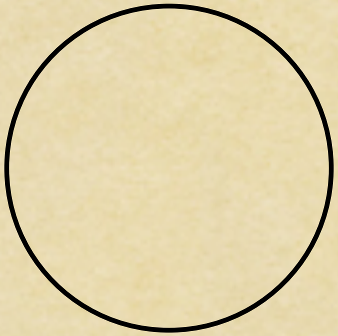



We call the family $\varphi_t$ (or the family $\varphi_t(K)$) an isotopy from $K_0 = \varphi_0(K) = K$ to $K_1 = \varphi_1(K)$, and say that the knots $K_0$ & $K_1$ are isotopic.

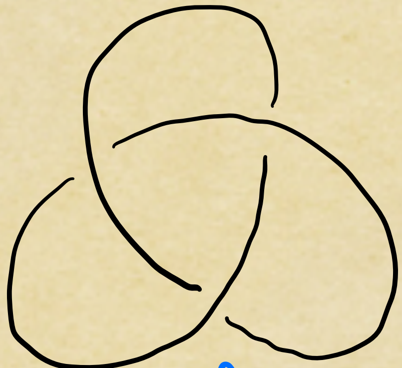
Basic classification problem: Catalogue all knots in $S^3$ up to isotopy

# Knots in $S^3$: definitions & examples

## Examples



unknot



trefoil

(left-handed)

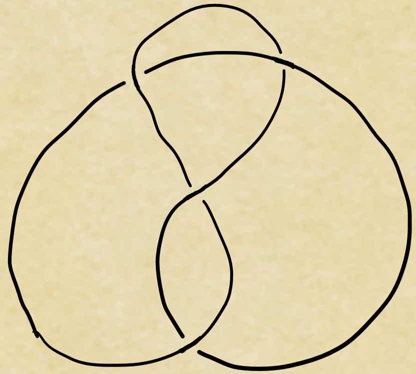
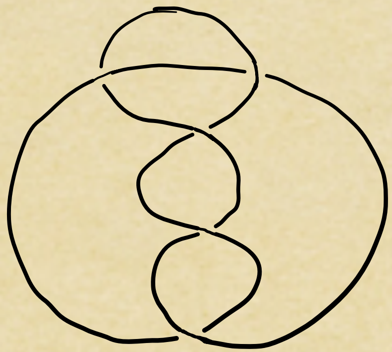
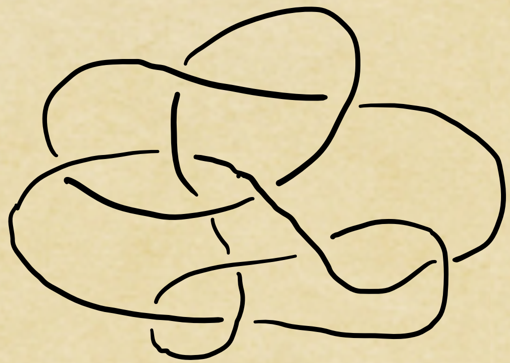


figure eight $= 4_1$

unknot / trefoil



$m(5_2)$



Conway knot

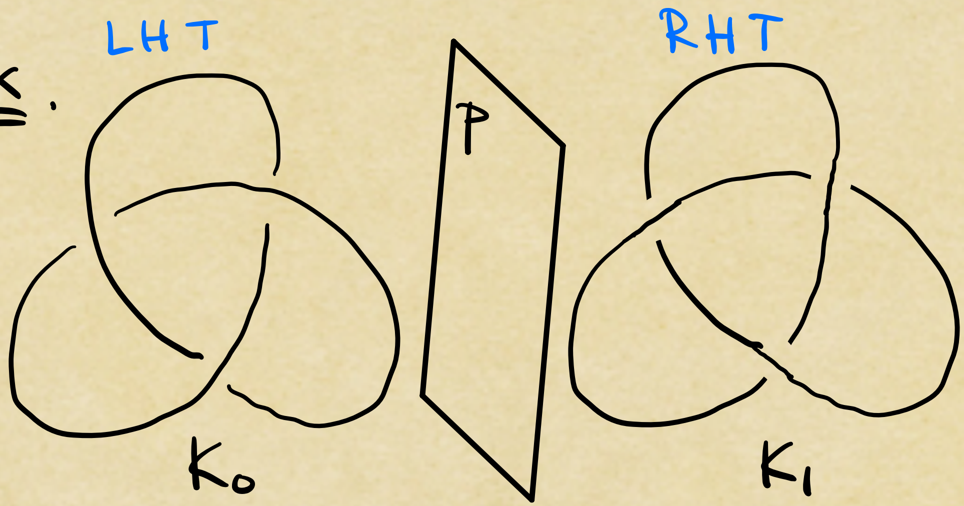
# Isomorphisms & isotopies

An isomorphism between knots only demands the last homeomorphism in the family $\varphi_t$. That is, we call $K_0$ & $K_1$ isomorphic if there is a homeomorphism $\varphi : S^3 \to S^3$ such that $K_1 = \varphi(K_0)$.

Whenever $K_0$ & $K_1$ are isotopic, we see that they must also be isomorphic: simply let $\varphi = \varphi_1$.

Reversing this implication seems difficult:

Ex.




We can obtain $K_1$ from $K_0$ by reflecting thru P.

Are $K_0$ & $K_1$ isotopic? No.

## Isomorphisms & isotopies

Here's a deep fact we won't prove:

**Thm.** Two knots $K_0, K_1 \subset S^3$ are isotopic if and only if they are isomorphic via an orientation-preserving homeo. $\varphi: S^3 \to S^3$.

Carefully defining what it means for a homeomorphism to be orientation-preserving involves some algebraic topology. But if $\varphi$ is diff'able and has $D_p\varphi \neq 0$ $\forall p \in S^3$, it just means that $\det(D_p\varphi) > 0$.

**Ex.** Reflecting a knot $K \subset S^3$ across a plane gives the mirror knot $m(K)$. But reflection is orientation-reversing, so $m(K)$ is isomorphic to $K$, but might not be isotopic.

$m(LHT) = RHT$ & $m(4_1) = 4_1$.