Math 4803 **April 1, 2024**

RECENTLY

Tessellations and quotient surfaces, mostly constructed via edge gluings of polygons.

TODAY

We'll start constructing the same things with a more "top-down" approach, using group actions by isometries.

Definitions & Examples

Recall that, for any set X, the collection of bijections $\widehat{\Gamma}_X = \{\gamma : X \to X \mid \gamma \text{ is bijective}\}$ forms a group under composition.

identity element: $\begin{aligned} id_X : X &\longrightarrow X \\ x &\longmapsto x \end{aligned}$

inverse of $\gamma \in \widehat{\Gamma}_X$: $\begin{aligned} \gamma^{-1} : X &\longrightarrow X \\ x &\longmapsto \gamma^{-1}(x) \end{aligned}$

A <u>transformation group</u> on a set X is a subgroup of $\widehat{\Gamma}_X$.

An equivalent notion is that of a <u>group action</u>, which is a map $\begin{aligned} \Gamma \times X &\longrightarrow X \\ (\gamma, x) &\longmapsto \gamma . x \end{aligned}$, where Γ is a group, s.t.

(a) $1.x = x$, $\forall x \in X$;

(b) $\gamma.(\phi.x) = (\gamma\phi).x$, $\forall \gamma, \phi \in \Gamma$ & $x \in X$.

subgroup

For any $\Gamma \leq \widehat{\Gamma}_X$ we can define $\Gamma \circlearrowright X$ by $\gamma . x = \gamma(x)$.

Definitions & Examples

We'll be most interested in isometric actions $\Gamma \curvearrowright (X,d)$, where (X,d) is a metric space and every element of Γ is an isometry $X \to X$.

(0) $\{id_X\} \curvearrowright (X,d)$

Ex. (1) The isometry group of a metric space is the collection $\mathrm{Isom}(X,d) \subset \widehat{\Gamma}_X$ of isometries $(X,d) \to (X,d)$.

Check: It's a subgroup of $\widehat{\Gamma}_X$.

(2) $\mathrm{Isom}(\mathbb{H}^2, d_{hyp}) = \text{LFMs} \cup \text{ALFMs}$

- LFMs form a subgroup of $\mathrm{Isom}(\mathbb{H}^2, d_{hyp})$
- ALFMs don't, since this set doesn't contain id_X

(It's also not closed under $\circ$.)

- The subset $\left\{ \dfrac{az+b}{cz+d} \;\middle|\; \begin{array}{l} ad-bc=1 \\ \& \ a,b,c,d \in \mathbb{Z} \end{array} \right\}$ is a subgroup of LFMs.

(It's isomorphic to $PGL(2, \mathbb{Q})$.)

Quotient spaces

Given a group of isometries Γ acting on a metric space (X,d), we can consider the orbit of any point $P \in X$:

$$\overline{P} = \{\gamma . P \mid \gamma \in \Gamma\} \subseteq X.$$

Lemma. The orbits of $\Gamma \curvearrowright (X,d)$ partition X.

(Proof.) Claim 1. $\bigcup_{P \in X} \overline{P} = X$

$\forall P \in X$, $P = 1.P \in \overline{P} \Rightarrow P \in \bigcup_{P \in X} \overline{P}$.

Claim 2. $\overline{P} \cap \overline{Q} \neq \phi \Rightarrow \overline{P} = \overline{Q}$.

$\S\, R \in \overline{P} \cap \overline{Q}$. Then $\exists\, \gamma, \phi \in \Gamma$ s.t. $\gamma . P = R = \phi . Q$

But then $P = \gamma^{-1}.(\phi . Q) = (\gamma^{-1} \circ \phi).Q$. Now consider any $S \in \overline{P}$. $\exists\, \eta \in \Gamma$ s.t. $S = \eta . P$, so

$$S = \eta . ((\gamma^{-1} \circ \phi).Q) = (\eta \circ \gamma^{-1} \circ \phi).Q \in \overline{Q}$$

Thus $\overline{P} \subseteq \overline{Q}$. Similarly, $\overline{Q} \subseteq \overline{P}$, so $\overline{P} = \overline{Q}$.

$\diamond$

Quotient spaces

The orbits of an action $\Gamma \circlearrowright (X,d)$ give a partition $\overline{X}$, which we may also denote as X/Γ. The various results we proved the first time we saw quotients give us a semi-metric $\overline{d}$ on $\overline{X}$. If $\overline{d}$ is an honest metric, we call $(\overline{X}, \overline{d})$ the quotient space or orbit space of $\Gamma \circlearrowright (X,d)$.

For quotients coming from group actions, we have a nice distance formula.

Lemma. Let (X,d) be a metric space, $\Gamma \circlearrowright (X,d)$ an action by a group of isometries. Then, $\forall \, \overline{P}, \overline{Q} \in \overline{X}$,

$$\overline{d}(\overline{P}, \overline{Q}) = \inf\{d(P', Q') \mid P' \in \overline{P}, Q' \in \overline{Q}\}$$
$$= \inf\{d(P, \gamma(Q)) \mid \gamma \in \Gamma\}$$

(Proof.) First, let's define

$$\overline{d}'(\overline{P}, \overline{Q}) := \inf\{d(P', Q') \mid P' \in \overline{P}, Q' \in \overline{Q}\}.$$

Quotient spaces

Lemma. Let (X,d) be a metric space, $\Gamma \curvearrowright (X,d)$ an action by a group of isometries. Then, $\forall\, \overline{P}, \overline{Q} \in \overline{X}$,

$$\overline{d}(\overline{P},\overline{Q}) = \inf\{ d(P',Q') \mid P' \in \overline{P},\ Q' \in \overline{Q} \}$$
$$= \inf\{ d(P,\gamma(Q)) \mid \gamma \in \Gamma \}.$$

(Proof, cont'd.) We'll show that $\overline{d} = \overline{d}'$.

Claim. $\overline{d}'$ satisfies the triangle inequality.

(Proof of claim.) Pick $\overline{P}, \overline{Q}, \overline{R} \in \overline{X}$ and $\varepsilon > 0$.

By def'n of infimum, $\exists\, P' \in \overline{P}$ & $Q' \in \overline{Q}$ s.t.

$$d(P',Q') < \overline{d}'(\overline{P},\overline{Q}) + \varepsilon/2.$$

Similarly, $\exists\, Q'' \in \overline{Q}$ & $R'' \in \overline{R}$ s.t.

$$d(Q'',R'') < \overline{d}'(\overline{Q},\overline{R}) + \varepsilon/2.$$

Now $\exists\, \gamma \in \Gamma$ s.t. $\gamma(Q') = Q''$. Let $R' = \gamma^{-1}(R'')$. Then

$$\overline{d}'(\overline{P},\overline{R}) \le d(P',R') \le d(P',Q') + d(Q',R')$$
$$= d(P',Q') + d(\gamma(Q'),\gamma(R'))$$

Quotient spaces

$$\begin{aligned}
\bar{d}'(\bar{P},\bar{R}) \le d(P',R') &\le d(P',Q') + d(Q',R') \\
&= d(P',Q') + d(\gamma(Q'),\gamma(R')) \\
&= d(P',Q') + d(Q'',R'') \\
&\le \bar{d}'(\bar{P},\bar{Q}) + \tfrac{\varepsilon}{2} + \bar{d}'(\bar{Q},\bar{R}) + \varepsilon/2
\end{aligned}$$

Since this holds $\forall\, \varepsilon > 0$, $\bar{d}'(\bar{P},\bar{R}) \le \bar{d}'(\bar{P},\bar{Q}) + \bar{d}'(\bar{Q},\bar{R})$. $\square$

Now suppose w is a discrete walk

$$P = P_1,\ Q_1 \sim P_2,\ Q_2 \sim P_3,\ \dots,\ Q_{n-1} \sim P_n,\ Q_n = Q$$

from P to Q. Then

triangle ineq.

$$\ell_d(w) = \sum_{i=1}^{n-1} d(P_i,Q_i) \ge \sum_{i=1}^{n-1} \bar{d}'(\bar{P}_i,\bar{Q}_i) \ge \bar{d}'(\bar{P}_1,\bar{Q}_n) = \bar{d}'(\bar{P},\bar{Q}).$$

Since $\bar{d}(\bar{P},\bar{Q}) = \inf\{\ell_d(w)\}$, we see that $\underline{\bar{d}(\bar{P},\bar{Q}) \ge \bar{d}'(\bar{P},\bar{Q})}$.

Finally, for any $P' \in \bar{P}$ and $Q' \in \bar{Q}$,

$$d(P',Q') \ge \bar{d}(\bar{P'},\bar{Q'}) = \bar{d}(\bar{P},\bar{Q})$$

But $\bar{d}'(\bar{P},\bar{Q}) = \inf\{d(P',Q')\}$, so $\underline{\bar{d}'(\bar{P},\bar{Q}) \ge \bar{d}(\bar{P},\bar{Q})}$.

Quotient spaces

So $\overline{d} = \overline{d}'$, where $\overline{d}'(\overline{P}, \overline{Q}) = \inf\{ d(P', Q') \mid P' \in \overline{P},\ Q' \in \overline{Q} \}$.

At last, let $\overline{d}''(\overline{P}, \overline{Q}) := \inf\{ d(P, \gamma(Q)) \mid \gamma \in \Gamma \}$.

Then $\overline{d}'(\overline{P}, \overline{Q}) = \overline{d}''(\overline{P}, \overline{Q})$, since

$$\{ d(P', Q') \mid P' \in \overline{P},\ Q' \in \overline{Q} \} = \{ d(\eta(P), \phi(Q)) \mid \eta, \phi \in \Gamma \}$$

$$= \{ d(P, (\eta^{-1} \circ \phi)(Q)) \mid \eta, \phi \in \Gamma \}$$

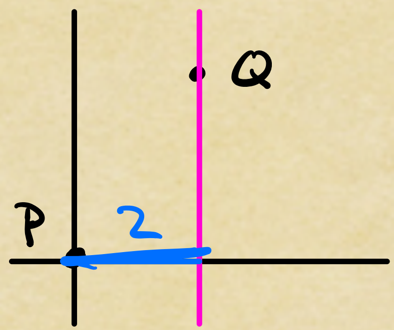
η^{-1} is an isometry

$$= \{ d(P, \gamma(Q)) \mid \gamma \in \Gamma \}.$$

So $\overline{d}(\overline{P}, \overline{Q}) = \overline{d}''(\overline{P}, \overline{Q}) = \inf\{ d(P, \gamma(Q)) \mid \gamma \in \Gamma \}$. $\Diamond$

Ex. $\mathbb{R} \curvearrowright (\mathbb{R}^2, d_{euc})$ by $t \cdot (x, y) = (x, y + t)$.

If $P = (0, 0)$ and $Q = (2, 3)$, then $\overline{d}(\overline{P}, \overline{Q}) = 2$



Discontinuous actions

We'll call an isometric action $\Gamma \curvearrowright (X,d)$ **discontinuous** if, for every $P \in X$, $\exists\, B_d(P,\varepsilon)$ s.t.

$$\{\gamma \in \Gamma \mid \gamma(P) \in B_d(P,\varepsilon)\}$$

is finite.

Ex. (1) The action $\mathbb{R} \curvearrowright (\mathbb{R}^2, d_{euc})$ given by $t.(x,y) = (x, y+t)$ is not discontinuous, but the restriction $\mathbb{Z} \curvearrowright (\mathbb{R}^2, d_{euc})$ is.

(2) For any integer $n \geq 1$, the action $\mathbb{Z}/n\mathbb{Z} \curvearrowright (\mathbb{R}^2, d_{euc})$ given by $n.(x,y) = (\cos(\tau/n)x - \sin(\tau/n)y, \sin(\tau/n)x + \cos(\tau/n)y)$ is discontinuous.

Thm. If $\Gamma \curvearrowright (X,d)$ is a discontinuous action by isometries, then $(\overline{X}, \overline{d})$ is a metric space.

Discontinuous actions

Thm. If $\Gamma \curvearrowright (X,d)$ is a discontinuous action by isometries, then $(\overline{X}, \overline{d})$ is a metric space.

(Proof.) All we need to check is that if $\overline{P} \neq \overline{Q}$, then $\overline{d}(\overline{P}, \overline{Q}) > 0$.

By the discontinuity of $\Gamma \curvearrowright (X,d)$, choose $\varepsilon > 0$ s.t.

$$\{\gamma \in \Gamma \mid \gamma(P) \in B_d(P, \varepsilon)\}$$

is finite and consider $\overline{Q} \cap B_d(P, \varepsilon/2)$.

Now $\overline{d}(\overline{P}, \overline{Q}) = \inf\{d(P, Q') \mid Q' \in \overline{Q}\}$, so if $\overline{Q} \cap B_d(P, \varepsilon/2) = \emptyset$, then $\overline{d}(\overline{P}, \overline{Q}) \geq \varepsilon/2 > 0$, so we're finished.

If $\overline{Q} \cap B_d(P, \varepsilon/2) \neq \emptyset$, then fix $Q_0 \in \overline{Q} \cap B_d(P, \varepsilon/2)$ and choose $\gamma_0 \in \Gamma$ s.t. $\gamma_0(Q) = Q_0$.

Any $Q' \in \overline{Q} \cap B_d(P, \varepsilon/2)$ gives $\gamma \in \Gamma$ s.t. $\gamma(Q) = Q'$ and we have: $(\gamma_0 \gamma_0^{-1})(P) \in B_d(P, \varepsilon)$ because

Discontinuous actions

Thm. If $\Gamma \curvearrowright (X,d)$ is a discontinuous action by isometries, then $(\overline{X}, \overline{d})$ is a metric space.

(Proof, cont'd).

$$d(P,(\gamma\circ\gamma_0^{-1})(P)) = d(\gamma^{-1}(P),(\gamma^{-1}\circ\gamma\circ\gamma_0^{-1})(P)) = d(\gamma^{-1}(P),\gamma_0^{-1}(P))$$
$$\leq d(\gamma^{-1}(P), Q) + d(Q,\gamma_0^{-1}(P)) \quad \text{(triangle inequality)}$$
$$= d(P,\gamma(Q)) + d(\gamma_0(Q), P)$$
$$< \varepsilon/2 + \varepsilon/2 = \varepsilon$$

Upshot: $(\gamma\circ\gamma_0^{-1})(P) \in B_d(P,\varepsilon)$, so there are finitely many options for γ, and thus for Q'. Enumerate these as

$$\{Q_1, Q_2, \ldots, Q_n\} = \overline{Q} \cap B_d(P,\varepsilon/2),$$

and notice that $d(P,Q_i) > 0$, for $i=1,\ldots,n$. b/c $P \neq Q_i$ & d is a metric

Then
$$\overline{d}(\overline{P},\overline{Q}) = \inf\{d(P,Q') \mid Q' \in \overline{Q}\}$$
$$= \min\{d(P,Q_i) \mid 1 \leq i \leq n\} > 0.$$

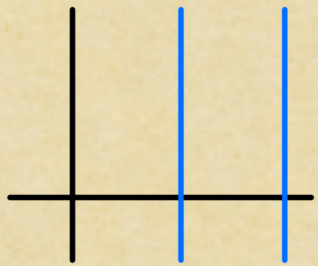
$\diamond$

Discontinuous actions

Thm. If $\Gamma \curvearrowright (X,d)$ is a discontinuous action by isometries, then $(\overline{X}, \overline{d})$ is a metric space.

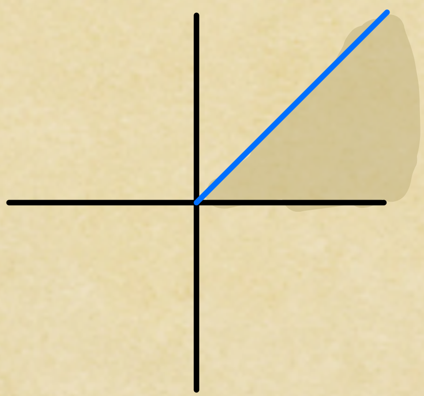
Ex. Let's describe the metric spaces which result from the discontinuous actions above.

(1) $\mathbb{Z} \curvearrowright (\mathbb{R}^2, d_{euc})$ given by $n.(x,y) = (x+n, y)$



$\rightsquigarrow$ "vertical" cylinder

(2) $\mathbb{Z}/n\mathbb{Z} \curvearrowright (\mathbb{R}^2, d_{euc})$ given by $n.(x,y) = R_{\tau/n}(x,y)$



$\rightsquigarrow$ Cone with cone angle $\tau/8$

Discontinuous actions

For any group action $\Gamma \curvearrowright X$ and element $P \in X$, the stabilizer of P is $\Gamma_P := \{\gamma \in \Gamma \mid \gamma(P) = P$. Check: $\Gamma_P \leq \Gamma$.

For an isometric action $\Gamma \curvearrowright (X,d)$, each $\gamma \in \Gamma_P$ satisfies $\gamma(B_d(P,\varepsilon)) = B_d(P,\varepsilon)$, for every $\varepsilon > 0$, and thus we get an isometric action $\Gamma_P \curvearrowright B_d(P,\varepsilon)$.

Thm. If $\Gamma \curvearrowright (X,d)$ is a discontinuous action by isometries, then for every $\overline{P} \in \overline{X}$, there is a ball $B_{\overline{d}}(\overline{P},\varepsilon) \subset \overline{X}$ which is isometric to the quotient space $B_d(P,\varepsilon)/\Gamma_P =: \widehat{B}_d(P,\varepsilon)$.

(Proof.) We'll use $\hat{d}$ to denote the quotient metric on $\widehat{B}_d(P,\varepsilon)$. Our goal is to construct an isometry $\varphi : \widehat{B}_d(P,\varepsilon) \to B_{\overline{d}}(\overline{P},\varepsilon)$, for some $\varepsilon > 0$. Since $\Gamma \curvearrowright (X,d)$ is discontinuous, $\exists\, \varepsilon > 0$ s.t. $d(P,\gamma(P)) < 5\varepsilon \Rightarrow \gamma(P) = P$.

Discontinuous actions

With this choice of $\varepsilon > 0$ we can define

$$\varphi : \hat{B}_d(P,\varepsilon) \longrightarrow B_{\bar{d}}(\bar{P},\varepsilon)$$
$$\hat{Q} \longmapsto \bar{Q}.$$

This is well-defined: if $\hat{R} = \hat{Q}$, then $\exists\, \gamma \in \Gamma_P$ s.t. $\gamma(Q) = R$. But $\Gamma_P \leq \Gamma$, so $\gamma \in \Gamma$ and thus $\bar{R} = \bar{Q}$.

Now for any $Q, R \in B_d(P,\varepsilon)$, we want to show that

$$\hat{d}(\hat{Q},\hat{R}) = \inf\{d(Q,\gamma(R)) \mid \gamma \in \Gamma_P\} = \inf\{d(Q,\gamma(R)) \mid \gamma \in \Gamma\} = \bar{d}(\bar{Q},\bar{R}).$$

When computing $\bar{d}(\bar{Q},\bar{R})$, the only values in $\{d(Q,\gamma(R)) \mid \gamma \in \Gamma\}$ that matter are those close to the infimum.

Claim: If $d(Q,\gamma(R)) \leq \bar{d}(\bar{Q},\bar{R}) + \varepsilon$, then $\gamma \in \Gamma_P$.

Discontinuous actions

Claim: If $d(Q,\gamma(R)) \le \bar{d}(\bar{Q},\bar{R})+\varepsilon$, then $\gamma \in \Gamma_P$.

(Proof of claim.) We use the triangle inequality:

$$\begin{aligned}
d(P,\gamma(P)) &\le d(P,Q) + d(Q,\gamma(R)) + d(\gamma(R),\gamma(P)) \\
&\le d(P,Q) + \bar{d}(\bar{Q},\bar{R}) + \varepsilon + d(R,P) \\
&\le d(P,Q) + d(Q,R) + \varepsilon + d(R,P) \\
&\le d(P,Q) + d(Q,P) + d(P,R) + \varepsilon + d(R,P) \\
&< 5\varepsilon .
\end{aligned}$$

Our assumption on ε then ensures that $\underline{\gamma(P)=P}$. $\boxtimes$

Upshot: $\bar{d}(\bar{Q},\bar{R}) = \hat{d}(\hat{Q},\hat{R})$.

Note that this immediately proves the <u>injectivity</u> of φ:

$$\varphi(\bar{Q}) = \varphi(\bar{R}) \Rightarrow \hat{d}(\varphi(\bar{Q}),\varphi(\bar{R})) = 0 \Rightarrow \bar{d}(\bar{Q},\bar{R}) = 0 \Rightarrow \bar{Q} = \bar{R}.$$

Finally, $\bar{Q} \in B_{\bar{d}}(\bar{P},\varepsilon) \Rightarrow \exists\, R \in \bar{Q}$ s.t. $d(P,R) < \varepsilon \Rightarrow \varphi(\hat{R})$ <u>makes sense.</u> But $\varphi(\hat{R}) = \bar{R} = \bar{Q}$, so φ is surjective. $\diamond$

Discontinuous actions

Finally, we can build Euclidean/hyperbolic/spherical surfaces if $\Gamma \curvearrowright (X,d)$ is free. We call an action $\Gamma \curvearrowright X$ free if $\Gamma_P = \{id_X\}$, for every $P \in X$.

Ex. In $(\mathbb{R}^2, d_{euc})$ & $(\mathbb{H}^2, d_{hyp})$, translation produces a free action, while rotation does not. Which actions (by isometries) on (S^2, d_{sph}) are free?

Cor. If $\Gamma \curvearrowright (X,d)$ is a discontinuous, free action by isometries, then $(\overline{X}, \overline{d})$ is locally isometric to (X,d).

(Proof.) We need to show that, for every $P \in X$, $\exists\ \varepsilon > 0$ s.t. $B_{\overline{d}}(\overline{P}, \varepsilon)$ is isometric to $B_d(P,\varepsilon)$. But

$$B_{\overline{d}}(\overline{P}, \varepsilon) \cong B_d(P,\varepsilon)/\Gamma_P = B_d(P,\varepsilon)/\{id\} \cong B_d(P,\varepsilon),$$

for some $\varepsilon > 0$. $\diamond$

Next

A connection between isometric group actions and tessellations.