Math 4441 September 7, 2022

LAST TIME

Curves in $\mathbb{R}^3$ are uniquely determined by their curvature and torsion, up to an isometry.

They're uniquely determined by the Frenet-Serret apparatus, up to translation.

TODAY

First, planar Frenet-Serret. Then, instead of local properties of space curves, let's think about global properties of planar curves.

Frenet-Serret in 2D

Remember what the Frenet frame gives us in 3D: an ONB at each point which is adapted to the curve.

Specifically, computing $\{\vec{T}', \vec{N}', \vec{B}'\}$ in terms of $\{\vec{T}, \vec{N}, \vec{B}\}$ gave us two functions which determine our curve, up to an isometry.

Unsurprisingly, we can build a 2D version.

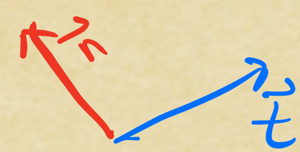
Frenet-Serret in 2D

Let $\vec{\alpha}(s)$ be a unit-speed $C^{\overset{2?}{1}}$ planar curve.

- The unit tangent vector field for $\vec{\alpha}$ is

$$\vec{t}(s) := \vec{\alpha}\,'(s).$$

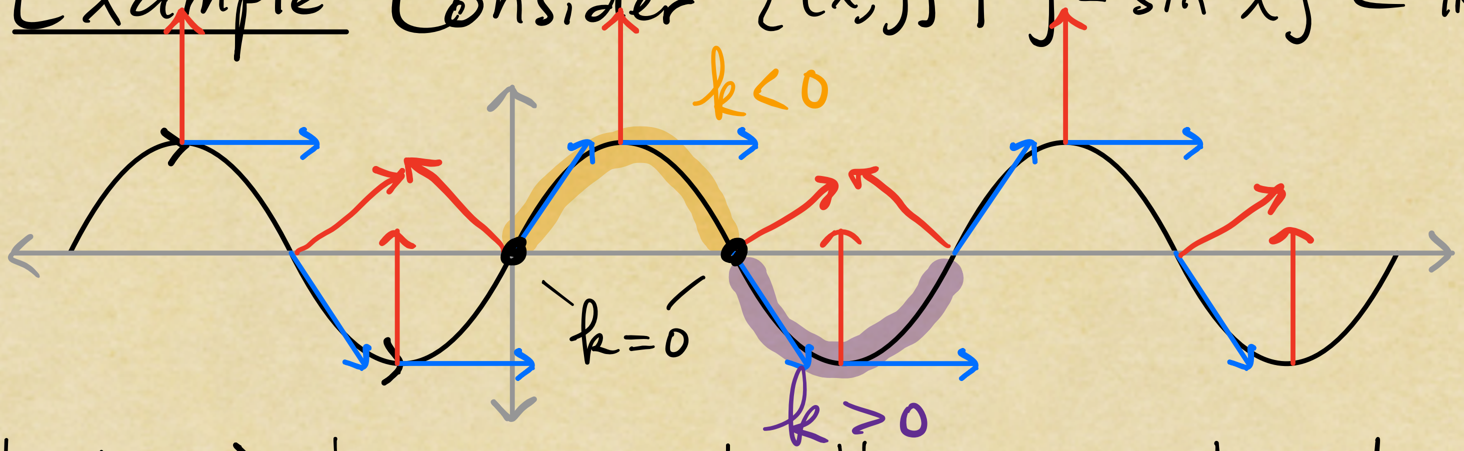
- The unit normal vector field for $\vec{\alpha}$ is defined by the requirement that $\underline{\{\vec{t}(s), \vec{n}(s)\}}$ is a <u>right-handed</u> ONB for $\mathbb{R}^2$, for all s.



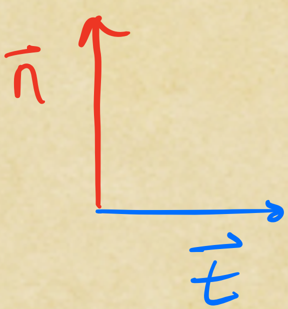
- The planar curvature $k(s)$ of $\vec{\alpha}(s)$ is defined by

$$k(s) := \langle \vec{t}\,'(s), \vec{n}(s) \rangle.$$

Example Consider $\{(x,y) \mid y=\sin x\} \subset \mathbb{R}^2$:



Let $\vec{\alpha}$ be an arclength parametrization of this curve. Let's plot $\vec{t}$ & $\vec{n}$ at some points, and determine the sign of k.



leftward turns are positive

Frenet-Serret in 2D

- We call $\{k, \vec{t}, \vec{n}\}$ the (planar) Frenet-Serret apparatus of $\vec{\alpha}(s)$.
- The planar curvature is sometimes called the signed curvature.
- The planar curvature determines $\vec{\alpha}(s)$ up to a choice of $\vec{\alpha}(0)$ and $\vec{t}(0)$. Equivalently, up to a rotation followed by a translation — i.e., an isometry of $\mathbb{R}^2$.

Formulas

Lemma. Let $\vec{\alpha}(s) = (x(s), y(s))$ be a unit-speed planar curve. Then

(a) $\vec{t}(s) = (x'(s), y'(s))$, $\vec{n}(s) = (-y'(s), x'(s))$

$$k(s) = x'(s)y''(s) - x''(s)y'(s)$$

(b) $$\begin{pmatrix} \vec{t}\,' \\ \vec{n}\,' \end{pmatrix} = \begin{pmatrix} 0 & k \\ -k & 0 \end{pmatrix} \begin{pmatrix} \vec{t} \\ \vec{n} \end{pmatrix}$$

(Proof.) Exercise.

Changing gears...

The fundamental theorem of space curves (and its 2D equivalent) basically reduces the study of curves to the study of functions (plus isometries)

But there are still interesting global quantities to consider. This will require recalling some vector calculus.

A vague rule of thumb:

the primary tool from calculus for local geometry is differentiation

for global properties, the main tool is integration

(also, topology tends to live on the global side)

Recollections

Def. Let $\vec{\alpha}:(a,b)\to\mathbb{R}^2$ be a planar curve, written as $\vec{\alpha}(t)=(x(t),y(t))$, and let $\vec{F}:\mathbb{R}^2\to\mathbb{R}^2$ be a vector field, written as $\vec{F}(x,y)=(f(x,y),g(x,y))$.

Then the line integral of $\vec{F}$ along $\vec{\alpha}$ is

$$\int_{\vec{\alpha}}\vec{F}\cdot d\vec{\alpha}:=\int_a^b\vec{F}(\vec{\alpha}(t))\cdot\vec{\alpha}'(t)\,dt$$

This is sometimes written as $\int_{\vec{\alpha}} f\,dx+g\,dy$.

Note: Assume $\vec{\alpha}$ & $\vec{F}$ are both, say, C^1.

If $\lambda : \mathbb{R}^2 \to \mathbb{R}$ is a scalar-valued function, we define <u>the line integral of λ over $\vec{\alpha}$</u> to be

$$\int_{\vec{\alpha}} \lambda\, ds := \int_a^b \lambda(\vec{\alpha}(t))\,|\vec{\alpha}'(t)|\, dt. \qquad ds = |\vec{\alpha}'(t)|\, dt$$

<u>Prop</u>. Line integration is a <u>geometric invariant</u> of curves, with a possible sign change for line integrals of vector fields. i.e., if $\vec{\beta}$ is a reparametrization of $\vec{\alpha}$, then

$$\int_{\vec{\alpha}} \vec{F} \cdot d\vec{\alpha} = \pm \int_{\vec{\beta}} \vec{F} \cdot d\vec{\beta} \quad ; \quad \int_{\vec{\alpha}} \lambda\, ds = \int_{\vec{\beta}} \lambda\, ds$$

(Proof.) Exercise using the chain rule. ◇

Example Let $\vec{F} = \langle x, y \rangle$ & $\vec{G} = \langle -y, x \rangle$.

Compute $\int_{\mathcal{C}} \vec{F} \cdot d\mathcal{C}$ & $\int_{\mathcal{C}} \vec{G} \cdot d\mathcal{C}$, where $\mathcal{C}$ is the circle of radius $R>0$ centered at the origin, with CCW orientation.

Note: We weren't given a param. for $\mathcal{C}$.

$$\vec{\alpha}(t) = (R\cos t, R\sin t) \rightarrow \vec{\alpha}'(t) = (-R\sin t, R\cos t)$$

$$\vec{F}(\vec{\alpha}(t)) = (R\cos t, R\sin t)$$

$$\vec{G}(\vec{\alpha}(t)) = (-R\sin t, R\cos t)$$

Example, cont'd

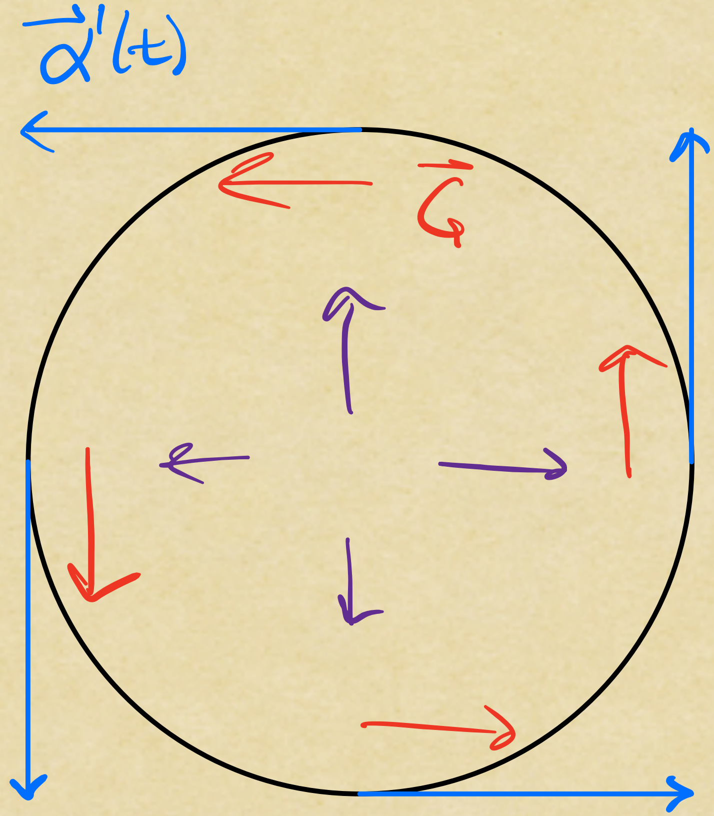
$$\int_C \vec{F} \cdot d\vec{\alpha} = \int_0^{2\pi} \vec{F}(\vec{\alpha}(t)) \cdot \vec{\alpha}'(t)\, dt$$

$$= \int_0^{2\pi} \left(-R^2 \cos t \sin t + R^2 \cos t \sin t\right) dt = 0$$

$$\int_C \vec{G} \cdot d\vec{\alpha} = \int_0^{2\pi} G(\vec{\alpha}(t)) \cdot \vec{\alpha}'(t)\, dt$$

$$= \int_0^{2\pi} R^2\, dt = 2\pi R^2$$

Interpretation



$\vec{G}$ pushes along $\vec{\alpha}$

$\vec{F} \perp$ the curve

Fundamental Theorems of Integral Calculus

We'll need two versions of the FTC:

Fundamental Theorem of Line Integrals

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a scalar-valued C^2 function, and let $\vec{\alpha} : I \to \mathbb{R}^n$ be a curve, with $[a,b] \subset I$. Then

$$\int_{\vec{\alpha}|_{[a,b]}} \nabla f \cdot d\vec{\alpha} = f(\vec{\alpha}(b)) - f(\vec{\alpha}(a)).$$

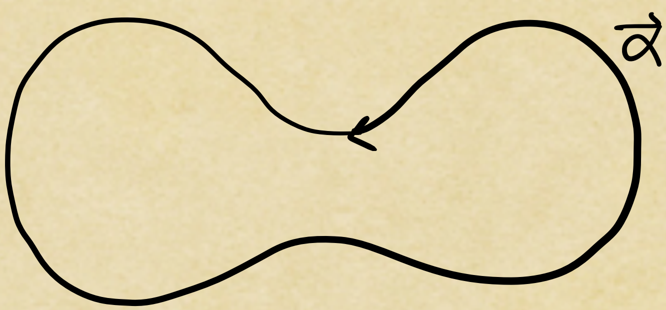
gradient of f is a v.f.

Green's theorem

Let $\vec{\alpha}$ be a closed, planar, C^2 curve which bounds a region R in a counterclockwise manner. Then for any $\vec{F}: \mathbb{R}^2 \to \mathbb{R}^2$,

$$\oint_{\vec{\alpha}} \vec{F} \cdot d\vec{\alpha} = \iint_R (\operatorname{curl} \vec{F} \cdot \vec{e}_3)\, dA,$$

where $\vec{F}$ is treated like a 3D vector field.



Next week: Using integral calculus to understand global properties of planar curves.