Math 4441 **September 28, 2022**

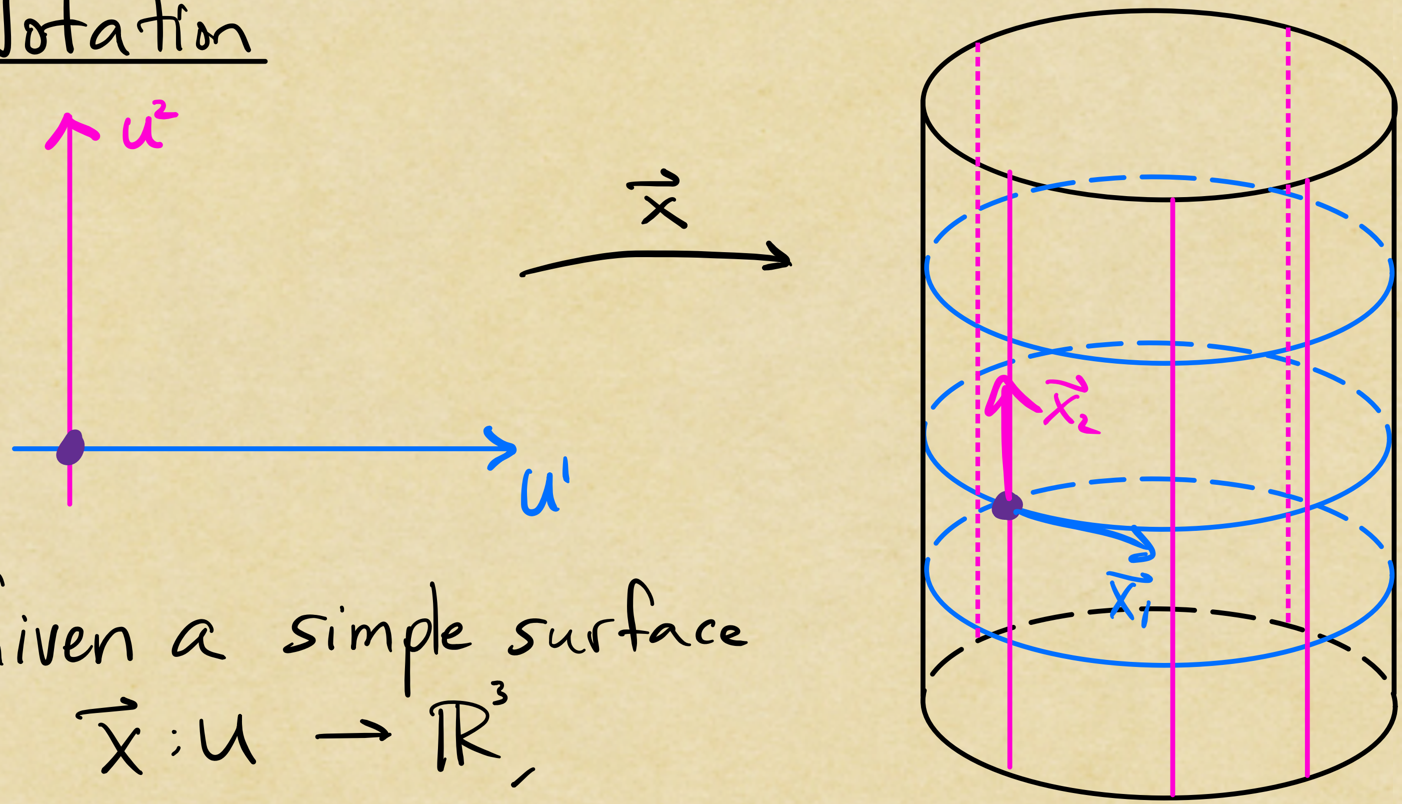
LAST TIME

As with curves, surfaces are functions, considered up to reparametrization

TODAY Our first geometric invariants.

curves	surfaces
$(a, b) \to \mathbb{R}^3$	$U \subset \mathbb{R}^2 \to \mathbb{R}^3$, with U open
reparam.	Coordinate transformations
tangent line	tangent plane

Notation



Given a simple surface

$$\vec{x} : U \to \mathbb{R}^3,$$

we have vectors

$$\vec{x}_1 := \frac{\partial \vec{x}}{\partial u^1} \quad \& \quad \vec{x}_2 := \frac{\partial \vec{x}}{\partial u^2}$$

These are the <u>columns</u> of $d\vec{x}$.

<u>Reparametrization</u> If $f : \tilde{U} \to U$ is a coordinate transformation and $\vec{\tilde{x}} := \vec{x} \circ f$, then the chain rule says

$$d\vec{\tilde{x}}_{\tilde{p}} = d\vec{x}_{f(\tilde{p})} \cdot df_{\tilde{p}}$$

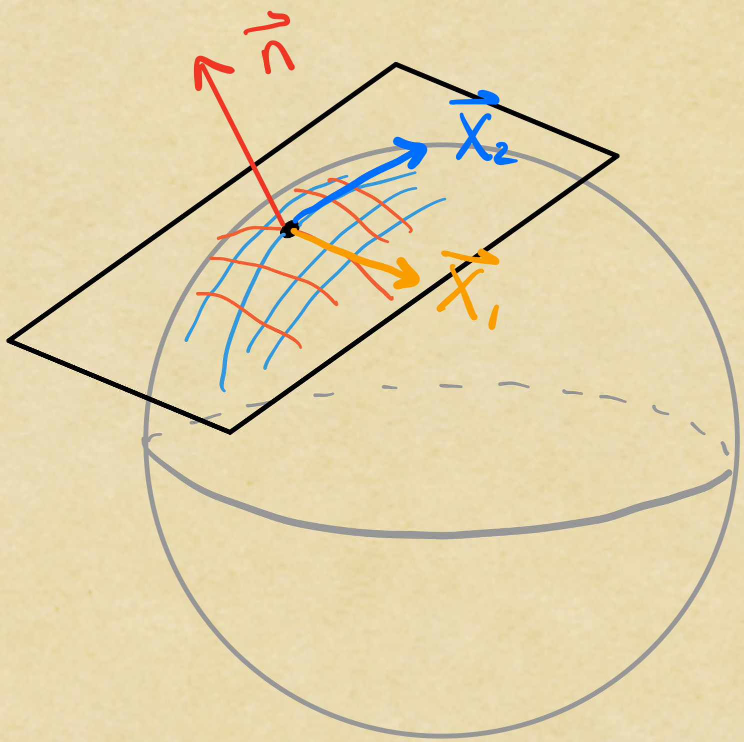
Writing out entries,

$$\begin{pmatrix} | & | \\ \vec{\tilde{x}}_1 & \vec{\tilde{x}}_2 \\ | & | \end{pmatrix} = \begin{pmatrix} | & | \\ \vec{x}_1 & \vec{x}_2 \\ | & | \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial f^1}{\partial u^1} & \frac{\partial f^1}{\partial u^2} \\ \frac{\partial f^2}{\partial u^1} & \frac{\partial f^2}{\partial u^2} \end{pmatrix}$$

In particular,

$$\vec{\tilde{x}}_j = \sum_{i=1}^{2} \frac{\partial f^i}{\partial u^j} \vec{x}_i . \qquad (\star)$$

The vectors $\vec{x}_1$ and $\vec{x}_2$ ought to be tangent to our surface S.



We want to use them* to define a tangent plane to S at $p = \vec{x}(u,v)$.

Remember that a plane in $\mathbb{R}^3$ is determined by a point plus a normal vector.

* It's tempting to take the span of these vectors. Not today.

Let $\vec{x} : U \to \mathbb{R}^3$ be a simple surface. The <u>unit normal vector</u> to $\vec{x}$ is the function

$$\vec{n} : U \longrightarrow S^2 \subset \mathbb{R}^3$$

defined by

$$\vec{n}(u^1,u^2) := \frac{\vec{x}_1(u^1,u^2) \times \vec{x}_2(u^1,u^2)}{|\vec{x}_1(u^1,u^2) \times \vec{x}_2(u^1,u^2)|}.$$

With $\vec{x}$ and $\vec{n}$ as above, let $p = \vec{x}(a,b)$, for some $(a,b) \in U$. The <u>tangent plane</u> $t_p\vec{x}$ to $\vec{x}(U)$ at p is <u>the plane through p which is $\perp$ to $\vec{n}$.</u>

(temporary notation)

Proposition. The tangent plane is a geometric invariant of surfaces.

(Proof.) We need to check that $t_p\vec{x}$ unchanged when we apply a coordinate transformation.

Let $\vec{\tilde{x}} = \vec{x} \circ f$. It will be enough to show that $\vec{\tilde{n}} = \pm \vec{n}$.

In fact, we don't need to scale down to unit length — just check that

$$\vec{\tilde{x}}_1 \times \vec{\tilde{x}}_2 \parallel \vec{x}_1 \times \vec{x}_2$$

But we previously computed that

$$\vec{\tilde{x}}_1 = \sum_{i=1}^{2} \frac{\partial f^i}{\partial u^1} \vec{x}_i \quad ; \quad \vec{\tilde{x}}_2 = \sum_{j=1}^{2} \frac{\partial f^j}{\partial u^2} \vec{x}_j$$

So $\vec{\tilde{x}}_1 \times \vec{\tilde{x}}_2 = \left(\sum_{i=1}^{2} \frac{\partial f^i}{\partial u^1} \vec{x}_i \right) \times \left(\sum_{j=1}^{2} \frac{\partial f^j}{\partial u^2} \vec{x}_j \right)$

$$= \frac{\partial f^1}{\partial u^1} \cdot \frac{\partial f^2}{\partial u^2} \vec{x}_1 \times \vec{x}_2 + \frac{\partial f^2}{\partial u^1} \cdot \frac{\partial f^1}{\partial u^2} \vec{x}_2 \times \vec{x}_1$$

$$= \left(\frac{\partial f^1}{\partial u^1} \cdot \frac{\partial f^2}{\partial u^2} - \frac{\partial f^2}{\partial u^1} \cdot \frac{\partial f^1}{\partial u^2} \right) \vec{x}_1 \times \vec{x}_2 = (\det df) \vec{x}_1 \times \vec{x}_2.$$

Since f is a coordinate transformation, $\det df \neq 0$, so we win. $\diamond$

Note: Unlike arclength parametrizations of curves, we don't have a naturally preferred function for surfaces.

So we have to do more things by hand.