Math 4441 September 21, 2022

LAST TIME

We wrapped up our study of the global theory of planar curves.

TODAY

We begin working towards a local theory of surfaces in $\mathbb{R}^3$. The first step is to review/enhance differentiation for functions of several variables.

Recall our approach to curves:

1. Use functions to represent curves, so that we can do calculus.
2. Let curves talk to each other via reparametrizations. These rely on heavy use of the chain rule.
3. Eventually, use integration to make global statements.

We'll apply the same philosophy to surfaces.*

* This story applies to manifolds of any dimension.

A surface is going to be a map

$$\vec{x} : U \to \mathbb{R}^3,$$

where $U \subset \mathbb{R}^2$ is an open domain, so we'll start by studying functions $\underline{f : \mathbb{R}^m \to \mathbb{R}^n}$ and their derivatives.

We write

$$\begin{array}{ccc} \mathbb{R}^m & \xrightarrow{\quad f \quad} & \mathbb{R}^n \\ p = (u^1, \dots, u^m) & \longmapsto & \begin{pmatrix} f^1(u^1, \dots, u^m) \\ f^2(u^1, \dots, u^m) \\ \vdots \\ f^n(u^1, \dots, u^m) \end{pmatrix} \end{array}$$

So $f(p) = \underline{(f^1(p), \dots, f^n(p))}$.

We call f C^k if each of $f^1, \dots, f^n$ is C^k.

If $f : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is C^1, then the derivative or differential of f, denoted df, is the matrix-valued function

$$df : \mathbb{R}^m \longrightarrow \mathrm{Mat}_{n \times m}(\mathbb{R})$$

defined by

$$df_p = \begin{pmatrix} \frac{\partial f^1}{\partial u^1}(p) & \cdots & \frac{\partial f^1}{\partial u^m}(p) \\ \vdots & \ddots & \vdots \\ \frac{\partial f^n}{\partial u^1}(p) & \cdots & \frac{\partial f^n}{\partial u^m}(p) \end{pmatrix},$$

where $u^1, \dots, u^m$ are the coordinates on $\mathbb{R}^m$.

Idea:

The derivative of a function f at a point p in $\mathrm{dom}(f)$ should give a linear approximation to f at p.

For a map $f: \mathbb{R}^m \to \mathbb{R}^n$, this means a linear map $L: \mathbb{R}^m \to \mathbb{R}^n$.

But this is precisely what is represented by an $n \times m$ matrix!

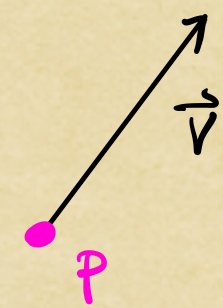
Vocab: We call df_p the Jacobian matrix of f at p.

Warning

The map $df : \mathbb{R}^m \to Mat_{n\times m}(\mathbb{R})$ is rarely linear!

Instead, df assigns a linear map $\mathbb{R}^m \to \mathbb{R}^n$ to each point of the domain.

Given a point $p \in \mathbb{R}^m$ and a vector $\vec{v} \in \mathbb{R}^m$, we think of $df_p(\vec{v})$ as measuring the change in f as we move in the direction* of $\vec{v}$ away from p



* at a speed dictated by $\vec{v}$

$df_p(\vec{v})$ is a vector.

$$df_p(c\cdot\vec{v}) = c\cdot df_p(\vec{v})$$

<u>Example</u> Consider the map $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ given by $f(u,v) = (u^2 + v,\ 4u + v^3 - 1)$.

Let's compute $df_{(0,0)}\begin{pmatrix}1\\1\end{pmatrix}$.

NOT LINEAR

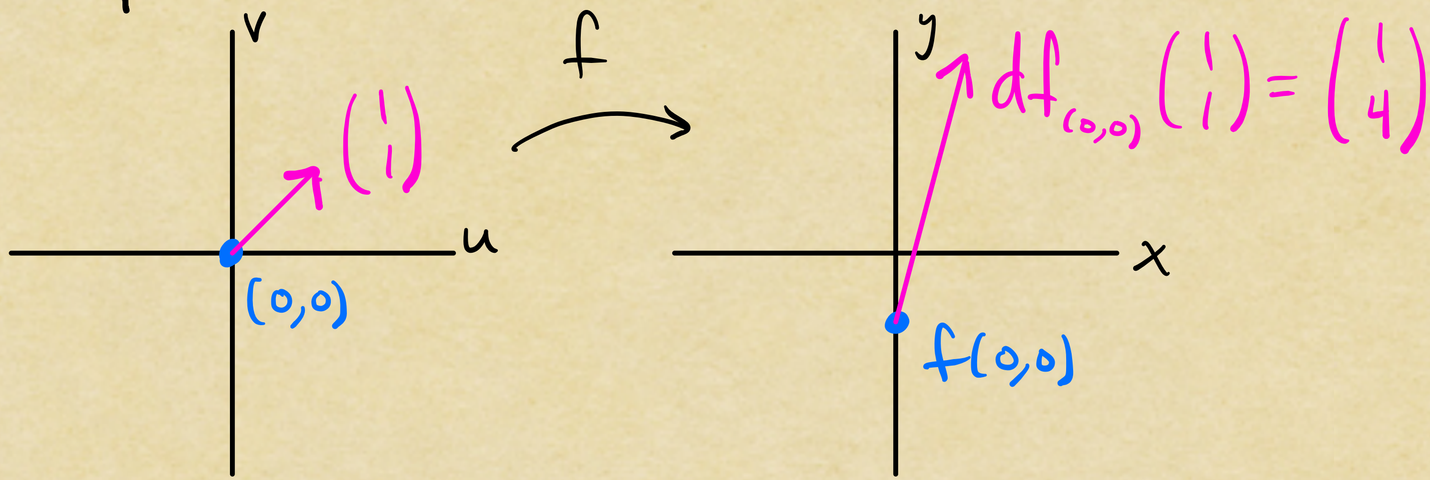
$$df_{(0,0)} = \begin{pmatrix} f^1_u(0,0) & f^1_v(0,0) \\ f^2_u(0,0) & f^2_v(0,0) \end{pmatrix} = \begin{pmatrix} 2u & 1 \\ 4 & 3v^2 \end{pmatrix}\Bigg|_{(0,0)} = \begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix}$$

LINEAR

So $df_{(0,0)}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\4\end{pmatrix}$

But! Where is this vector based?

Example, cont'd $\quad f(u,v) = (u^2 + v,\ 4u + v^3 - 1)$.



The notation $df_{(0,0)}\begin{pmatrix}1\\1\end{pmatrix}$ tells us that $\begin{pmatrix}1\\1\end{pmatrix}$ is based at $\underline{(0,0)}$. The image vector $\underline{df_p(\vec{v})}$ is then based at $\underline{f(0,0) = (0,-1)}$.

The chain rule

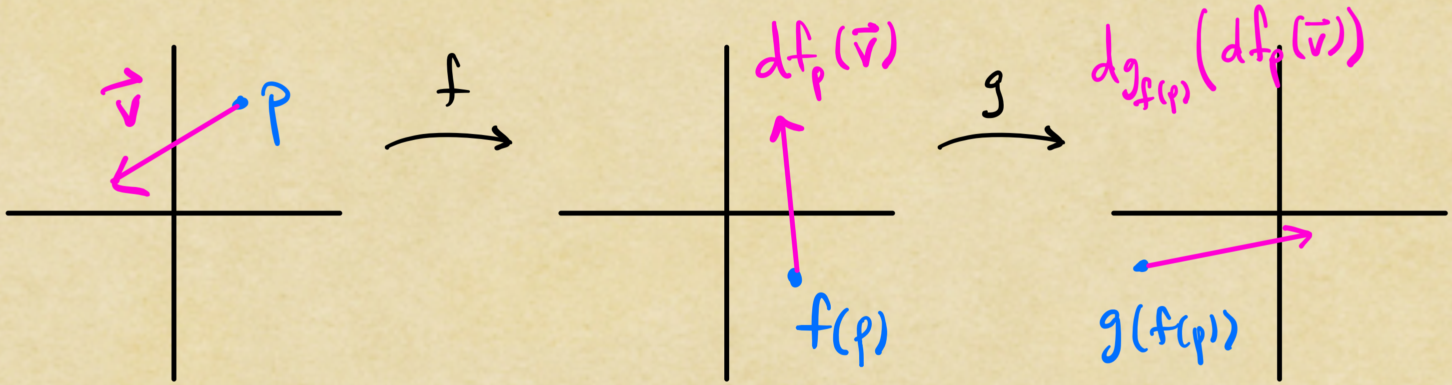
$$\frac{d}{dx}\big(g(f(x))\big) = g'(f(x))\cdot f'(x)$$

In words, the chain rule says that the derivative of a composition is the composition of the derivatives. This still works.

The chain rule If $f: \mathbb{R}^\ell \to \mathbb{R}^m$ and $g: \mathbb{R}^m \to \mathbb{R}^n$ are C^1 functions, then so is $g \circ f: \mathbb{R}^\ell \to \mathbb{R}^n$, and

$$d(g\circ f)_p = dg_{f(p)} \cdot df_p$$

for all $p \in \mathbb{R}^\ell$.



In coordinates

We have $f : \mathbb{R}^\ell \to \mathbb{R}^m$; $g : \mathbb{R}^m \to \mathbb{R}^n$.

Given coordinates $u^1, \ldots, u^\ell$ on $\mathbb{R}^\ell$ and $v^1, \ldots, v^m$ on $\mathbb{R}^m$, you'll work out an expression for $\dfrac{\partial (g \circ f)^i}{\partial u^j}$ in terms of $\dfrac{\partial g^i}{\partial v^k}$ and $\dfrac{\partial f^k}{\partial u^j}$, for $1 \le i \le n$, $1 \le j \le \ell$, ; $p \in \mathbb{R}^\ell$. We'll call this the indexed version of the chain rule.

Indexed version of the chain rule

Hopefully you found that

$$\left.\frac{\partial (g\circ f)^i}{\partial u^j}\right|_p = \sum_{k=1}^{m} \left.\frac{\partial g^i}{\partial v^k}\right|_{f(p)} \cdot \left.\frac{\partial f^k}{\partial u^j}\right|_p .$$

Note that this is the familiar chain rule for

$$g^i\left(f^1(u^1,\dots,u^\ell),\dots,f^m(u^1,\dots,u^\ell)\right),$$

but the matrices do better job of packaging.

We'll use both forms (matrix + indexed).

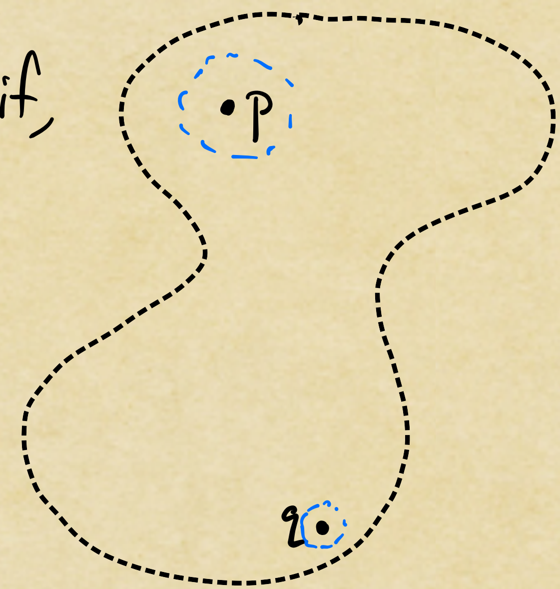
Simple surfaces

We're now almost ready to talk about surfaces, which we said we'd work with as functions

$$\vec{x} : U \to \mathbb{R}^3,$$

where U is an open subset of $\mathbb{R}^2$.

Recall that $U \subset \mathbb{R}^2$ is open if, for every $p \in U$, the open disc $D_\varepsilon(p)$ of radius ε centered at p is contained in U, for some $\varepsilon > 0$.



A C^k simple surface is an injective C^k function

$$\vec{x} : U \longrightarrow \mathbb{R}^3,$$

where $U \subset \mathbb{R}^2$ is open, such that the vectors

$$\frac{\partial \vec{x}}{\partial u^1} \text{ \& } \frac{\partial \vec{x}}{\partial u^2},$$

3×2 matrix

which form the columns of $d\vec{x}_p$, are linearly independent in $\mathbb{R}^3$, for all $p \in U$.

Note: This condition on $d\vec{x}_p$ corresponds to the regularity condition for curves.

Conditions for regularity

Lemma. An injective C^k function $\vec{x} : U \to \mathbb{R}^3$ is a simple surface iff, for every $p \in U$,

(a) the vectors $\frac{\partial \vec{x}}{\partial u^1}(p)$ & $\frac{\partial \vec{x}}{\partial u^2}(p)$ are lin. ind.;

iff

(b) the linear map $d\vec{x}_p : \mathbb{R}^2 \to \mathbb{R}^3$ has $\ker d\vec{x}_p = \{\vec{0}\}$;

iff

(c) $\frac{\partial \vec{x}}{\partial u^1}(p) \times \frac{\partial \vec{x}}{\partial u^2}(p) \neq \vec{0}$.

(Proof.) linear algebra $\diamond$

We'll see examples of simple surfaces soon, but first let's say what reparametrizations are for surfaces.

A C^k **coordinate transformation** is an invertible function $f : \tilde{U} \twoheadrightarrow U$ between open subsets $U, \tilde{U} \subset \mathbb{R}^2$ such that f and f^{-1} are C^k functions.

Theorem. If $f : \tilde{U} \to U$ is C^1 and bijective and satisfies $\det(df_{\tilde{p}}) \neq 0$ for all $\tilde{p} \in \tilde{U}$, then f is a C^1 coordinate transformation.

We'll try to give more intuition for simple surfaces and C^k coordinate transformations next time.