Math 4441 September 19, 2022

LAST TIME

The rotation index of a simple, closed, planar curve must be $\underline{\pm 1}$. Also, convexity.

TODAY

Our final planar curves question: <u>how much area does $\vec{\alpha}$ enclose?</u>

The isoperimetric problem

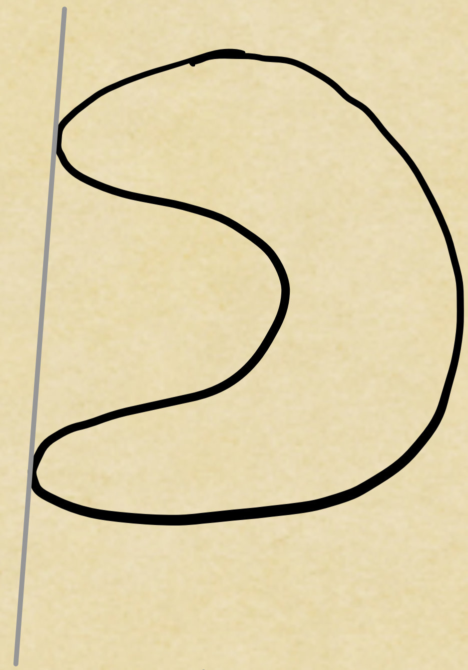
same perimeter

What is the maximum area that can be enclosed by a regular, planar curve of fixed length? Is there a unique curve which encloses this area?

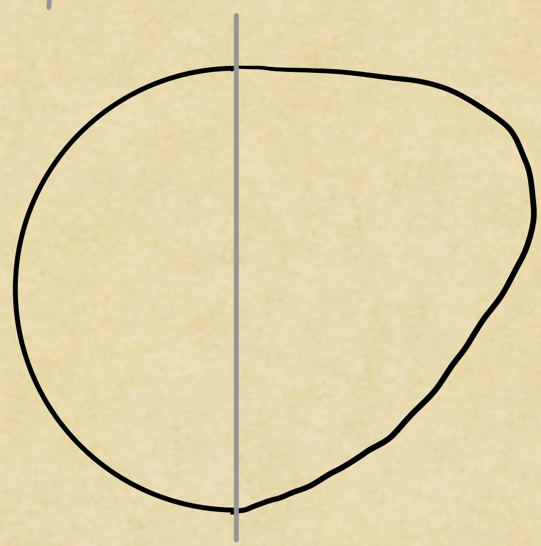
Circle

Some observations

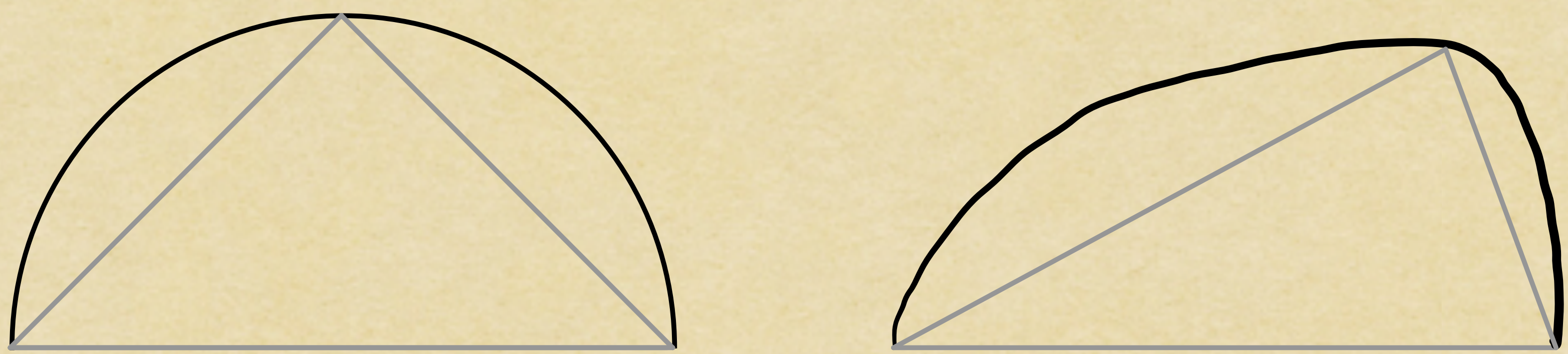
① Convexity If $\vec{\alpha}$ isn't convex, flip a portion out to increase area w/out changing perimeter.



② Symmetry If one side encloses a larger area, reflect it across to increase total area.



③ Circularity $\exists$ classical geometry argument which says each half is circular

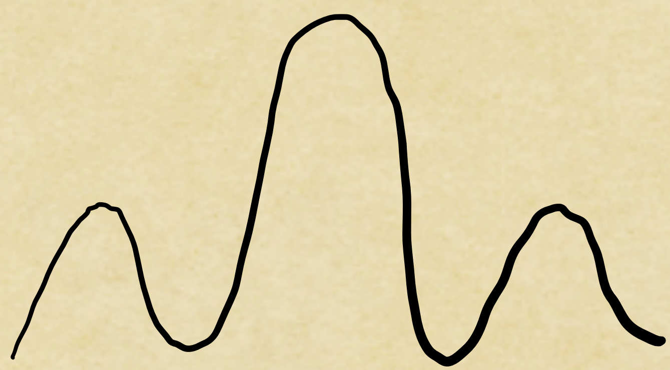


We'll skip it.

This argument is sometimes called Steiner's symmetrization.

While it arrives at the correct answer, this argument isn't considered to be fully rigorous by modern standards.

The main problem is that this argument assumes the existence of a unique solution.



Historical(ish) note

The problem of maximizing the area bounded by a fixed perimeter (and its solution) has been known since antiquity.

According to the Aeneid, Queen Dido used the solution to maximize the land area she could enclose with an oxhide.

Today we'll give a more rigorous proof of this fact.

<u>The isoperimetric inequality</u> Let $\vec{\alpha}$ be a simple, closed, regular, planar curve with perimeter L and enclosed area A. Then

$$A \le \frac{L^2}{4\pi},$$

with equality if and only if $\vec{\alpha}$ is a circle

We have three things to check:

1. any $\vec{\alpha}$ satisfies the inequality;
2. for circles we have equality;
3. if the inequality is an equality, $\vec{\alpha}$ must be a circle.

We won't check 2 today.

Our proof relies on three key facts:

(A) <u>An area formula.</u> For $\vec{\alpha}$ as above,

$$A = \int_{\vec{\alpha}} x\,dy = \int_{\vec{\alpha}} -y\,dx .$$

HW

(B) <u>The Cauchy–Schwarz inequality</u> For any $\vec{v}, \vec{w} \in \mathbb{R}^n$,

$$|\langle \vec{v}, \vec{w} \rangle| \leq |\vec{v}| \cdot |\vec{w}|$$

with equality if and only if $\underline{\vec{v} \parallel \vec{w}}$.

(C) <u>The AM–GM inequality</u> For any positive reals $a, b > 0$, $\underline{\sqrt{ab} \leq \frac{a+b}{2}}$, with equality if and only if $\underline{a = b}$

HW

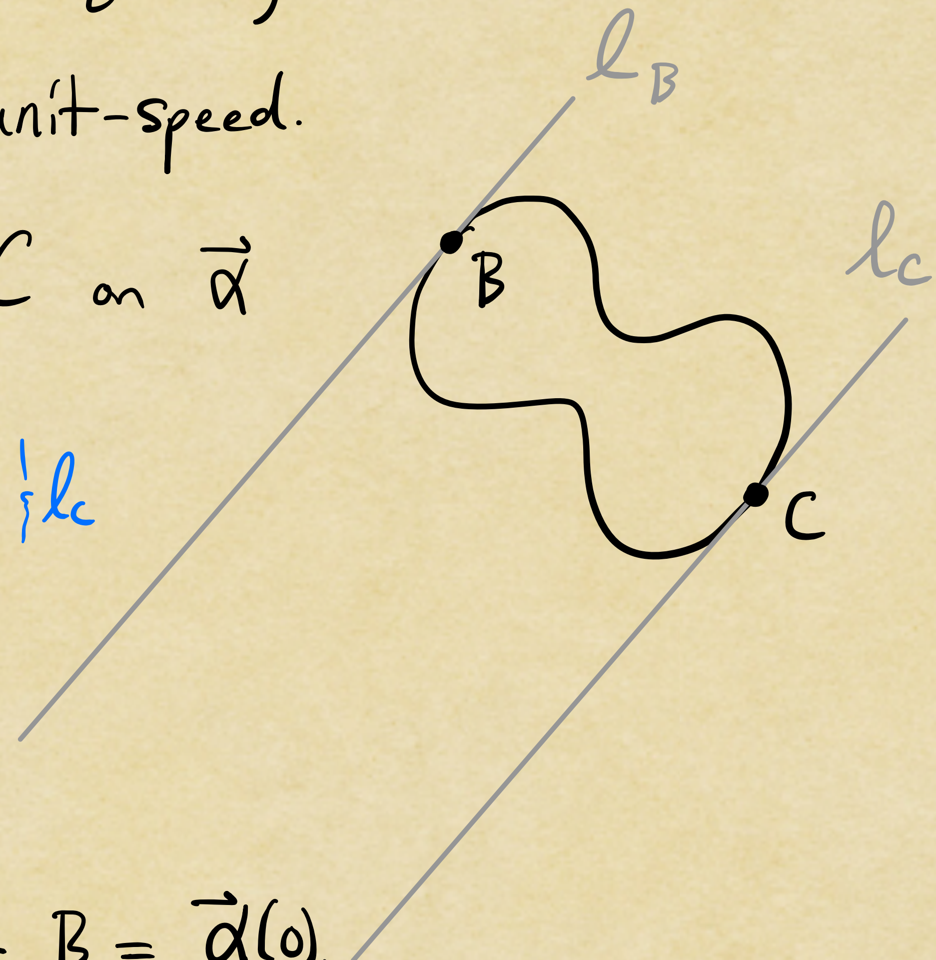
(Proof of the isoperimetric inequality)

① Establishing the inequality

Throughout, $\vec{\alpha}(s)$ is unit-speed.

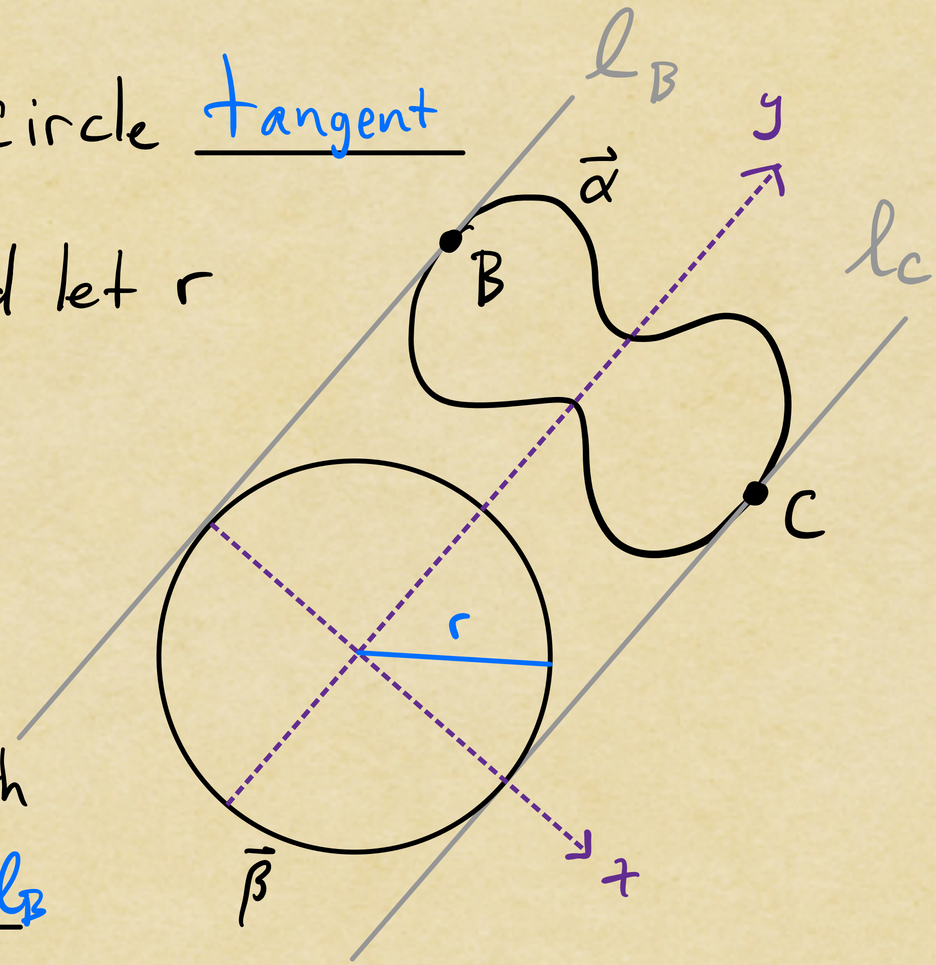
Choose points B and C on $\vec{\alpha}$ so that

- the tangent lines ℓ_B & ℓ_C are parallel;
- $\mathrm{im}(\vec{\alpha})$ lies entirely between these lines.



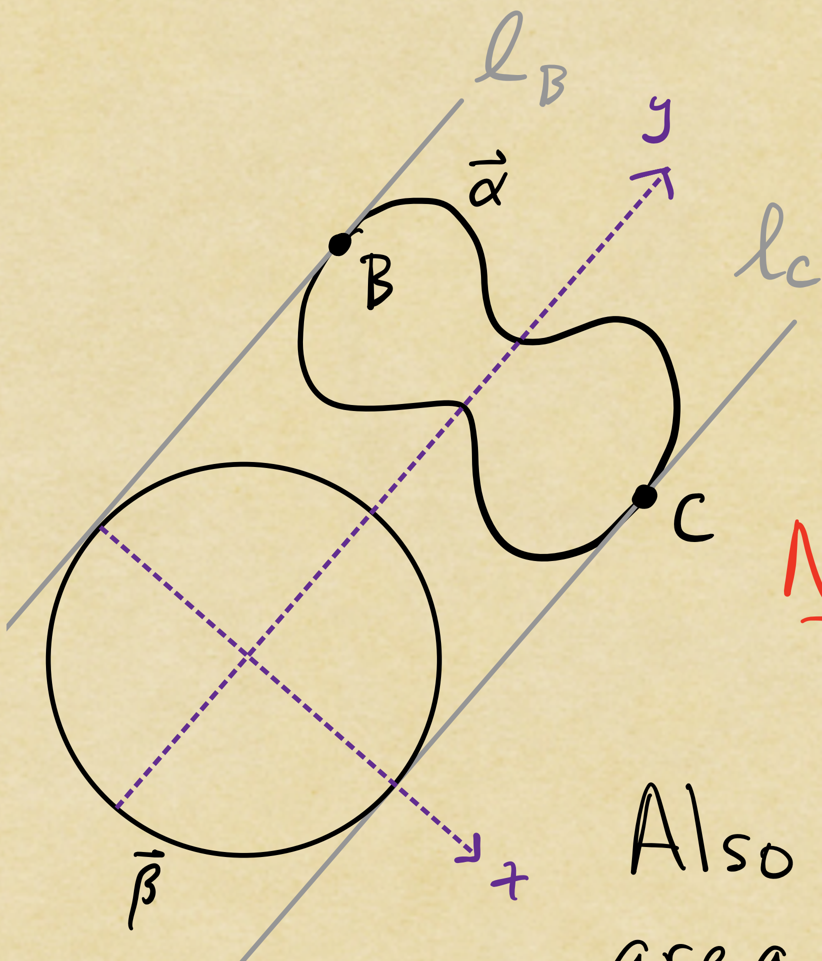
We can assume that $B = \vec{\alpha}(0)$.

Now we let $\vec{\beta}$ be any circle tangent to both ℓ_B & ℓ_C, and let r denote its radius.



Take coordinates (x,y) with origin at the center of $\vec{\beta}$, and with the y-axis parallel to ℓ_B & ℓ_C.

In these coords, write $\vec{\alpha} = \begin{pmatrix} x(s) \\ y(s) \end{pmatrix}$.



In this coordinate system, we can parametrize $\vec{\beta}$ as

$$\vec{\beta}(s) = \begin{pmatrix} x(s) \\ w(s) \end{pmatrix}$$

<u>Note</u>: This param. might not be <u>regular</u>! (NBD)
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Also note that $\vec{\beta}$ encloses an area of $\underline{\pi r^2}$.

So, from fact (A):

$$A = \int_{\vec{\alpha}} x\, dy \quad \& \quad \pi r^2 = \int_{\vec{\beta}} -w\, dx$$

Time to compute:

$$A + \pi r^2 = \int_{\vec{\alpha}} x\,dy - \int_{\vec{\beta}} w\,dx$$

$$= \int_0^L \left(x(s)\,y'(s) - w(s)\,x'(s) \right) ds$$

$$= \int_0^L \left\langle \begin{pmatrix} -w(s) \\ x(s) \end{pmatrix}, \begin{pmatrix} x'(s) \\ y'(s) \end{pmatrix} \right\rangle ds$$

$$\overset{(*)}{\leq} \int_0^L \left| \left\langle \begin{pmatrix} -w(s) \\ x(s) \end{pmatrix}, \begin{pmatrix} x'(s) \\ y'(s) \end{pmatrix} \right\rangle \right| ds$$

$$\leq \int_0^L \left\| \begin{pmatrix} -w(s) \\ x(s) \end{pmatrix} \right\| \cdot \left\| \begin{pmatrix} x'(s) \\ y'(s) \end{pmatrix} \right\| ds$$

$$= \int_0^L \sqrt{(w(s))^2 + (x(s))^2} \cdot |\vec{\alpha}'(s)|\,ds = \int_0^L r \cdot 1\,ds = rL$$

So $\dfrac{A+\pi r^2}{2} \le \dfrac{rL}{2}$. On the other hand,

the AM-GM inequality gives

$$\sqrt{A\pi r^2} \le \frac{A+\pi r^2}{2} \le \frac{rL}{2} \rightarrow \sqrt{A\pi r^2} \le \frac{rL}{2}$$

Squaring both sides:

$$\therefore\ A\pi \cancel{r^2} \le \frac{\cancel{r^2} L^2}{4} \longrightarrow A \le \frac{L^2}{4\pi}$$

So the inequality is established.

③ All that's left is to check that $A = \frac{L^2}{4\pi}$ implies that $\vec{\alpha}$ is a circle.

We still have the double inequality:

$$\sqrt{A\pi r^2} \overset{\cancel{\leq}}{=} \frac{A + \pi r^2}{2} \overset{\cancel{\leq}}{=} \frac{rL}{2}$$

We need both parts to be equalities.

AM-GM: Equality implies $A = \pi r^2$

$$\therefore \pi r^2 = \frac{L^2}{4\pi} \rightarrow r = \frac{L}{2\pi}$$

Second part: In the long computation, we need equality in Cauchy-Schwarz:

$$\left| \left\langle \begin{pmatrix} -w \\ x \end{pmatrix}, \begin{pmatrix} x' \\ y' \end{pmatrix} \right\rangle \right| = \left\| \begin{pmatrix} -w \\ x \end{pmatrix} \right\| \cdot \left\| \begin{pmatrix} x' \\ y' \end{pmatrix} \right\| .$$

This second part is Cauchy-Schwarz, where we have equality iff the vectors are parallel

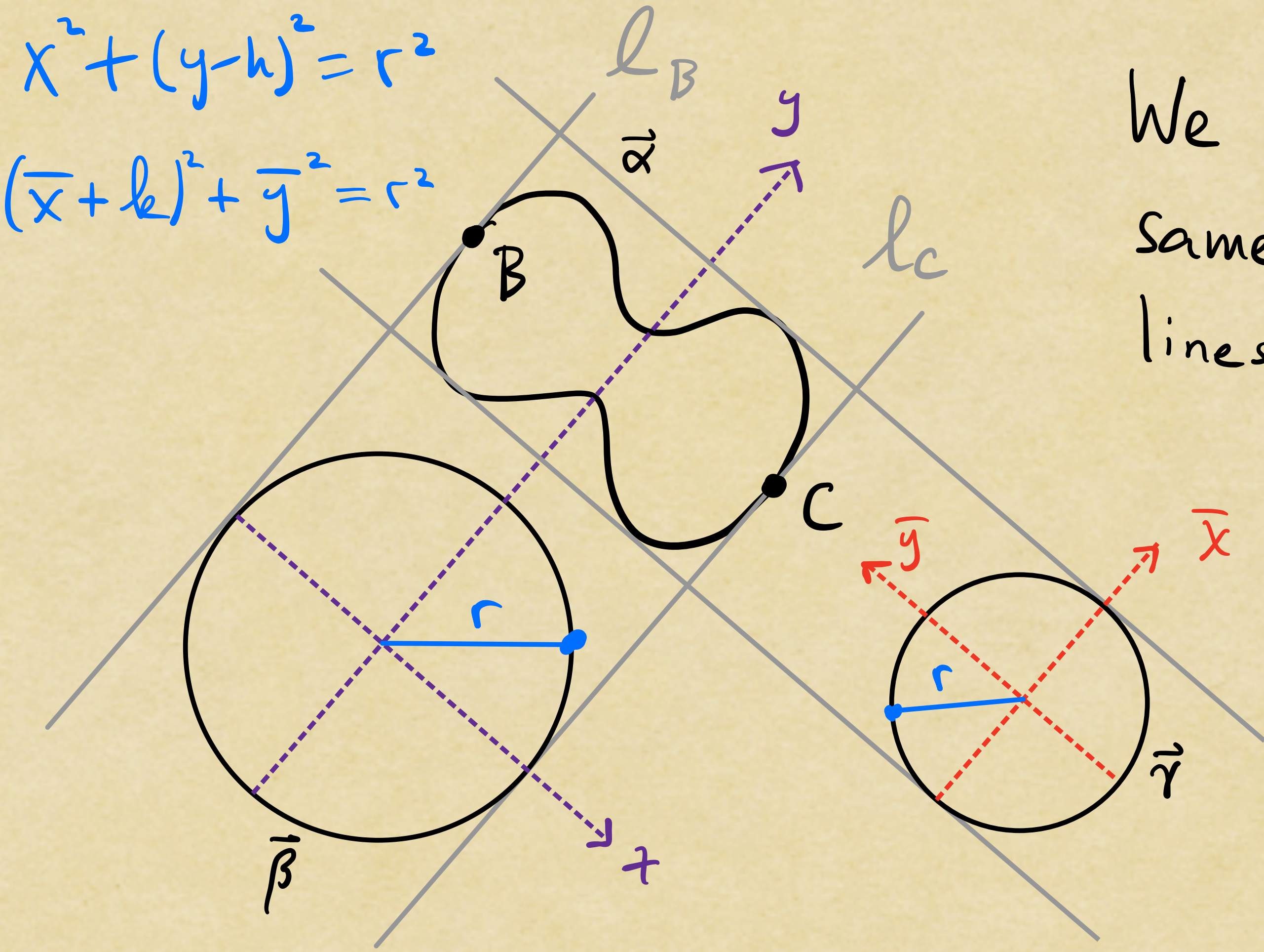
So $\begin{pmatrix} -w \\ x \end{pmatrix} = c \cdot \begin{pmatrix} x' \\ y' \end{pmatrix}$. In fact,

The circle $\vec{\beta}$ has param $\begin{pmatrix} x \\ w \end{pmatrix}$.

$$\left(\begin{array}{l} c \geq 0 \text{ to} \\ \text{make } (*) \\ \text{an } = \end{array}\right) c = \left| c \cdot \begin{pmatrix} x' \\ y' \end{pmatrix} \right| = \left| \begin{pmatrix} -w \\ x \end{pmatrix} \right| = r$$

So $-w = r \cdot x'$; $x = r \cdot y'$.

For the next step, we need to enhance our picture.



We can play the same game with lines $\perp$ to ℓ_B, ℓ_C.

We'll get a new circle, but still have $r = \frac{L}{2\pi}$.

The $\perp$ lines allow us to choose coords $(\bar{x},\bar{y})$ with $\bar{x} = $ <u>$y-h$</u>, $\bar{y} = $ <u>$k-x$</u>, for some constants h, k.

Recall: $\vec{\alpha}(s) = \begin{pmatrix} x(s) \\ y(s) \end{pmatrix}$. We also have $\vec{\alpha}(s) = \begin{pmatrix} \bar{x}(s) \\ \bar{y}(s) \end{pmatrix}$.

Finally, $x = r y'$ and $\bar{x} = r\bar{y}'$ mean that

$$\begin{aligned} x^2 + (y-h)^2 &= (ry')^2 + \bar{x}^2 \\ &= (ry')^2 + (r\bar{y}')^2 \\ &= r^2\left((y')^2 + (\bar{y}')^2\right) \\ &= r^2\left((y')^2 + (-x')^2\right) \\ &= r^2 \end{aligned}$$

$\bar{y} = k - x$

$\bar{y}' = -x'$

with the last equality using the fact that $\vec{\alpha}$ is <u>unit speed</u>. So $\vec{\alpha}$ must be a circle! $\diamond$

Conclusion

There's a unique curve with perimeter L which maximizes enclosed area, and this curve is a circle.