Math 4441 September 14, 2022

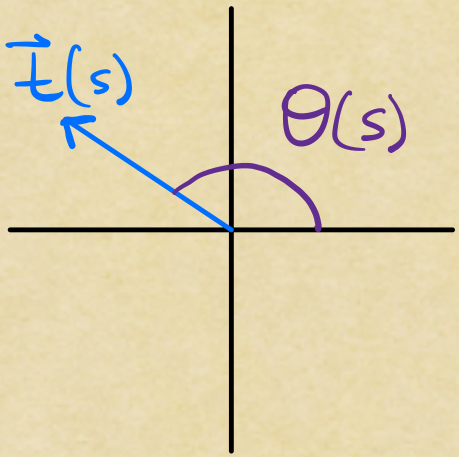
LAST TIME

We defined the rotation index of a closed, unit-speed curve. Along the way we defined an angular rotation function* θ.

TODAY

We'll prove the rotation index theorem, which basically says that the rotation index is what we want it to be.

What did we do, again?



We wanted $\theta(s)$ to satisfy

$$(\star) \quad \vec{t}(s) = \begin{pmatrix} \cos\theta(s) \\ \sin\theta(s) \end{pmatrix},$$

so that it measures rotation of $\vec{t}$.

Trying to define θ directly from $(\star)$ gives a discontinuous function. But we can use $(\star)$ to define θ', and then set

$$\theta(s) = \theta_0 + \int_0^s \theta'(t)\, dt.$$

This will be as smooth as $\vec{\alpha}(s)$.

VERY IMPORTANT OBSERVATION

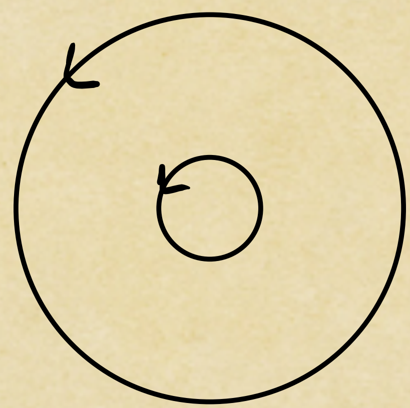
For a unit-speed $\vec{\alpha}(s) = (x(s), y(s))$, we defined $\theta(s)$ to be

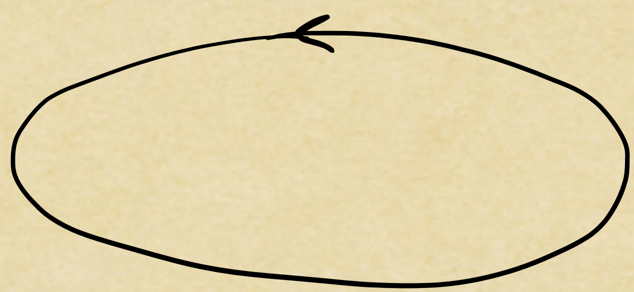
$$\theta(s) = \theta_0 + \int_0^s \big(x'(t)y''(t) - x''(t)y'(t)\big)\,dt .$$

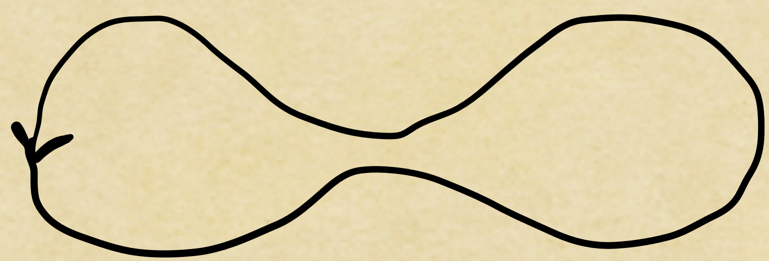
We've seen this before. It's the signed curvature of $\vec{\alpha}$! i.e., $\boxed{\theta' = k}$

So we obtain the rotation index of $\vec{\alpha}$ by integrating the signed curvature of $\vec{\alpha}$.

This will provide a bridge between geometry and topology.







Here's the important fact we'll prove today.

The Rotation Index Theorem

The rotation index of a simple, closed, planar curve is ± 1.

So the rotation index turns a continuous quantity (curvature) into a discrete quantity (the rotation index).

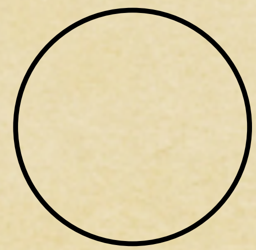
This might seem more impressive when we do it for surfaces.

(Proof) Let's start by outlining the key ideas.

We have

$$[0, L] \xrightarrow{\vec{\alpha}}$$

$$[0, L] \xrightarrow{\vec{t}}$$



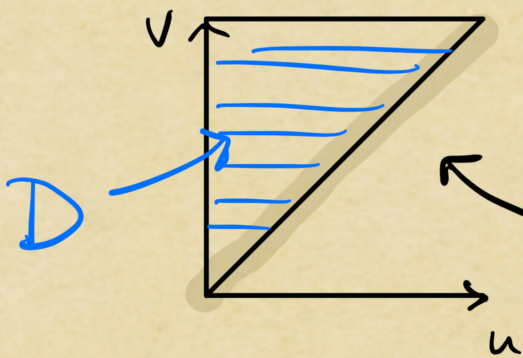
$= S^1$

$$\varepsilon \; [0, L] \xrightarrow{\theta} \mathbb{R}$$

and want $i_{\vec{\alpha}} = \dfrac{\theta(L) - \theta(0)}{2\pi}$.

We're going to enhance $\Theta : [0, L] \to \mathbb{R}$ to

$$\phi : \{(u,v) \mid 0 \le u \le v \le L\} \to \mathbb{R}.$$



$\xrightarrow{\phi} \mathbb{R}$

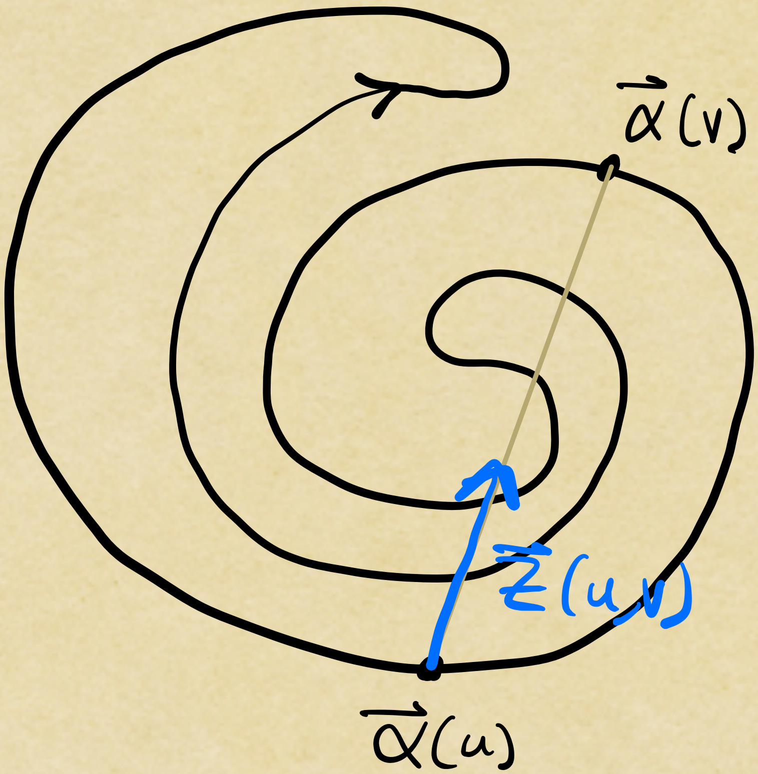
$\phi(u,u) = \Theta(u)$

Then $\Theta(L) - \Theta(0) = \underline{\phi(L,L) - \phi(0,0)}$,

so we can (hope to) use $\underline{\text{Green's theorem}}$.

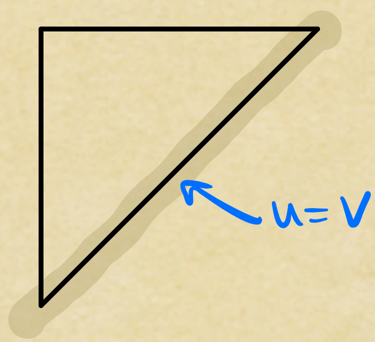
Step 1: Enhance $\vec{t} : [0, L] \longrightarrow S^1$ to

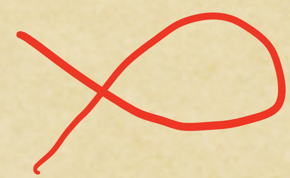
$$\vec{z} : D \longrightarrow S^1$$



Define

$$\vec{z}(u,v) := \begin{cases} \dfrac{\vec{\alpha}(v) - \vec{\alpha}(u)}{|\vec{\alpha}(v) - \vec{\alpha}(u)|}, & u < v \\ \vec{t}(u), & u = v \\ -\vec{t}(0), & u = 0, \ v = L \end{cases}$$





Note: For $\vec{z}$ to be continuous, it's crucial that $\vec{\alpha}$ is <u>simple</u>.

Step 2: Enhance $\theta : [0, L] \longrightarrow \mathbb{R}$ to

$\phi : D \longrightarrow \mathbb{R}$.

Let's write $\vec{z}(u,v) = \begin{pmatrix} f(u,v) \\ g(u,v) \end{pmatrix}$. Because $\vec{z}$ takes values in S^1, we can hope to write

$$\vec{z}(u,v) = \begin{pmatrix} \cos \phi(u,v) \\ \sin \phi(u,v) \end{pmatrix},$$

for some $\phi(u,v)$. Remember that

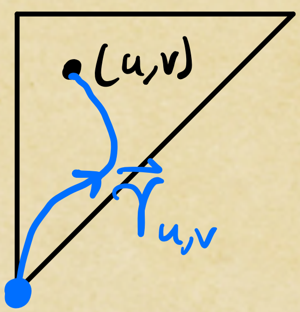
$$\vec{t}(s) = \begin{pmatrix} \cos \theta(s) \\ \sin \theta(s) \end{pmatrix},$$

and $\theta(s) = \theta_0 + \int_0^s \left(x'(t)y''(t) - x''(t)y'(t)\right) dt.$

We'll build $\phi(u,v)$ by imitating the construction of $\theta(s)$: it's a fixed angle plus an integral.

Let $\vec{\Psi}(u,v) := \begin{pmatrix} f \cdot g_u - f_u \cdot g \\ f \cdot g_v - f_v \cdot g \end{pmatrix}$.

We want to define $\phi(u,v) = \phi_0 + \int_{\vec{\gamma}_{u,v}} \vec{\Psi} \cdot d\vec{\gamma}_{u,v}$,



where $\vec{\gamma}_{u,v}$ is any path from (0,0) to (u,v) in D

This requires $\vec{\Psi}$ to be conservative.

Check : $\vec{\Psi}$ is conservative ($\equiv$ curl-free)

$$\operatorname{curl} \vec{\Psi} = \frac{\partial}{\partial u}\left(f \cdot g_v - f_v \cdot g\right) - \frac{\partial}{\partial v}\left(f \cdot g_u - f_u \cdot g\right)$$

You should verify that this vanishes.

Since $\vec{\Psi}$ is conservative, the path we use to get from $(0,0)$ to (u,v) doesn't matter.

So $$\phi(u,v) = \phi_0 + \int_{\vec{\gamma}_{u,v}} \vec{\Psi} \cdot d\vec{\gamma}_{u,v}$$

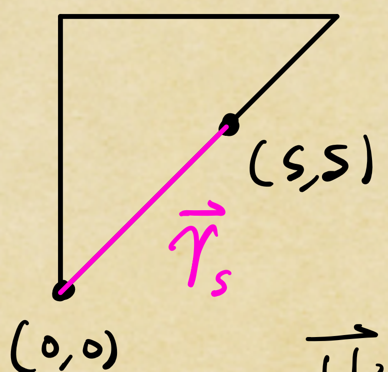
is well-defined.

(This is where we're using <u>Green's theorem</u>.)

We need to convince ourselves that this extends θ. That is,

$$\theta(s) = \phi(s,s),$$

for all $s \in [0, L]$.



Indeed, you can check that if $\vec{\gamma}_s$ is a unit-speed curve $(0,0) \rightsquigarrow (s,s)$, then

$$\vec{\psi}(\vec{\gamma}_s(t)) \cdot \vec{\gamma}_s{}'(t) = x'(t)y''(t) - x''(t)y'(t),$$

so

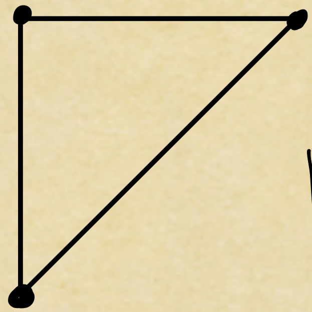
$$\phi(s,s) = \phi_0 + \int_0^s \vec{\psi}(\vec{\gamma}_s(t)) \cdot \vec{\gamma}_s{}'(t)\, dt$$

$$= \theta_0 + \int_0^s (x'(t)y''(t) - x''(t)y'(t))\,dt = \theta(s)$$

Step 3: Compute $2\pi i_{\vec{\alpha}} = \theta(L) - \theta(0)$

$$= \phi(L,L) - \phi(0,0)$$

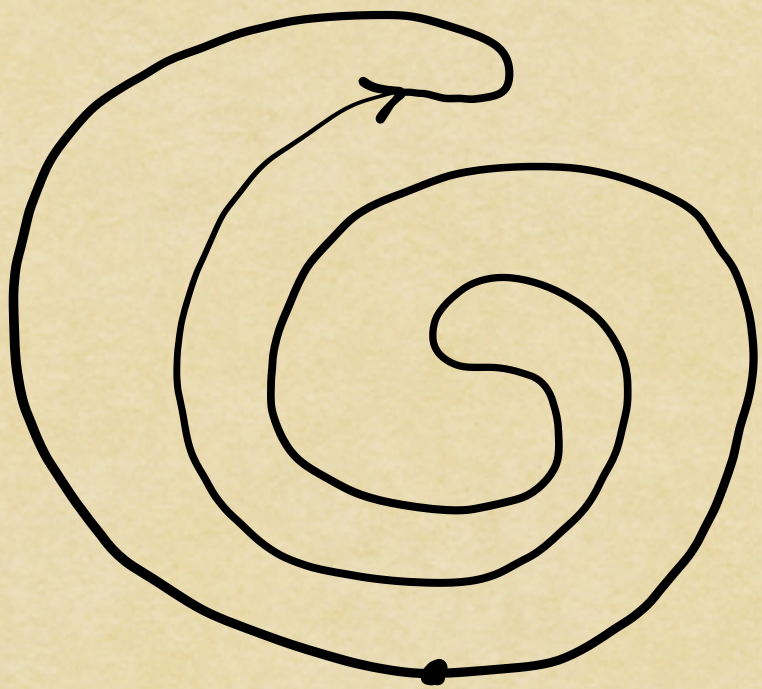
(0,L) (L,L)



(0,0)

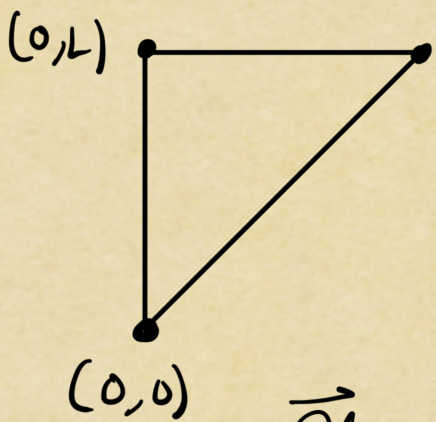
We'll be clever with our choice of path.

We'll also fix our parametrization so that $\vec{\alpha}(0)$ is the bottommost point on $\vec{\alpha}$.



$\vec{\alpha}(0)$

Upshot: $\vec{z}(0,u)$ never points downward.



Now

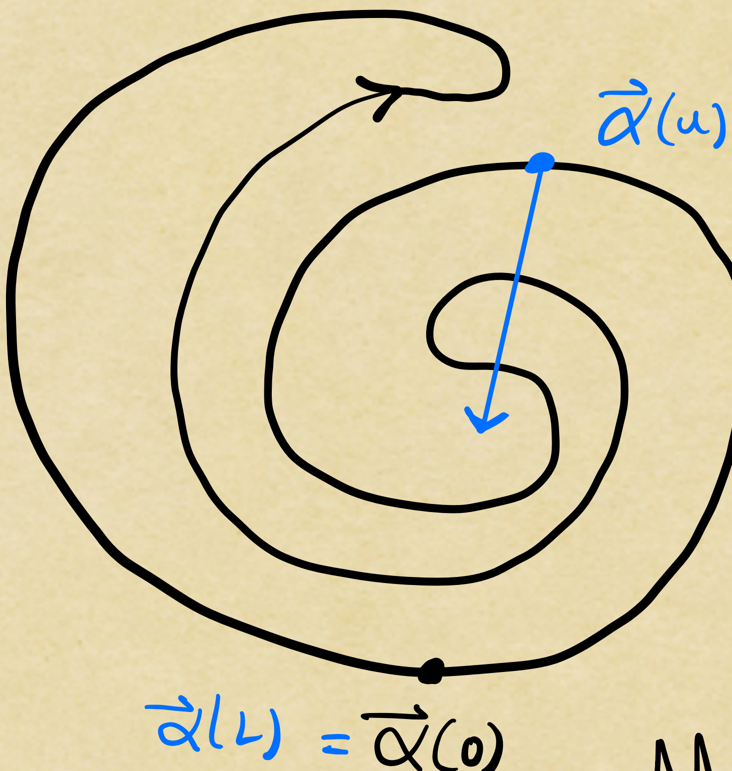
$$\Phi(L,L) = \Phi_0 + \int_{\vec{\gamma}_1} \vec{\Psi}\cdot d\vec{\gamma}_1 + \int_{\vec{\gamma}_2} \vec{\Psi}\cdot d\vec{\gamma}_2$$

$\vec{\gamma}_1$: $(0,0) \rightsquigarrow (0,L)$

$\vec{\gamma}_2$: $(0,L) \rightsquigarrow (L,L)$

So $\int_{\vec{\gamma}_1} \vec{\Psi}\cdot d\vec{\gamma}_1$ measures the angle change in $\vec{Z}(0,v)$ as v runs from 0 to L. Because $\vec{Z}(0,v)$ never points downward, this must be $\pm\pi$.

On the other hand, $\int_{\vec{\gamma}_2} \vec{\Psi} \cdot d\vec{\gamma}_2$ measures the angle change for $\underline{\vec{z}(u,L)}$ as u varies from 0 to L. This vector never points upward, so



$$\int_{\vec{\gamma}_2} \vec{\Psi} \cdot d\vec{\gamma}_2 = \underline{\pm\pi}.$$

Moreover,

$$\int_{\vec{\gamma}_2} \vec{\Psi} \cdot d\vec{\gamma}_2 = \int_{\vec{\gamma}_1} \vec{\Psi} \cdot d\vec{\gamma}_1.$$

Altogether,

$$\phi(L,L) = \phi_0 + \int_{\vec{\gamma}_1} \vec{\psi}\cdot d\vec{\gamma}_1 + \int_{\vec{\gamma}_2} \vec{\psi}\cdot d\vec{\gamma}_2$$

$$= \phi(0,0) \pm \pi \pm \pi \quad ,$$

so $\phi(L,L) - \phi(0,0) = \pm 2\pi$.

Finally, $$i_{\vec{\alpha}} = \frac{\theta(L) - \theta(0)}{2\pi}$$

$$= \pm 1 \quad ,$$

as desired. $\diamond$