Math 4441 September 12, 2022

LAST TIME

We began to focus on global properties of planar curves, and recalled some FTICs.

TODAY

Our first global property: the rotation index

Closed curves

The rotation index will only be defined for closed curves, so we need a definition:

A regular curve $\vec{\beta}(t)$ is closed if it's periodic as a function of t. i.e., there's some $T>0$ such that

$$\vec{\beta}(t+T) = \vec{\beta}(t)$$

for all t for which this makes sense. The smallest such T is the period of $\vec{\beta}$.

If $\vec{\beta}$ is a closed curve with period T, then the perimeter of $\vec{\beta}$ is defined by

$$L := \int_0^T |\vec{\beta}'(t)|\, dt$$

Lemma If $\vec{\beta}$ is a closed curve with perimeter L, and $\vec{\alpha}$ is an arclength reparametrization of $\vec{\beta}$, then $\vec{\alpha}$ is closed with period L

(Proof.) Exercise using linearity of integration

$\diamond$

One more preparation: what does it mean for a curve to have self-intersections?

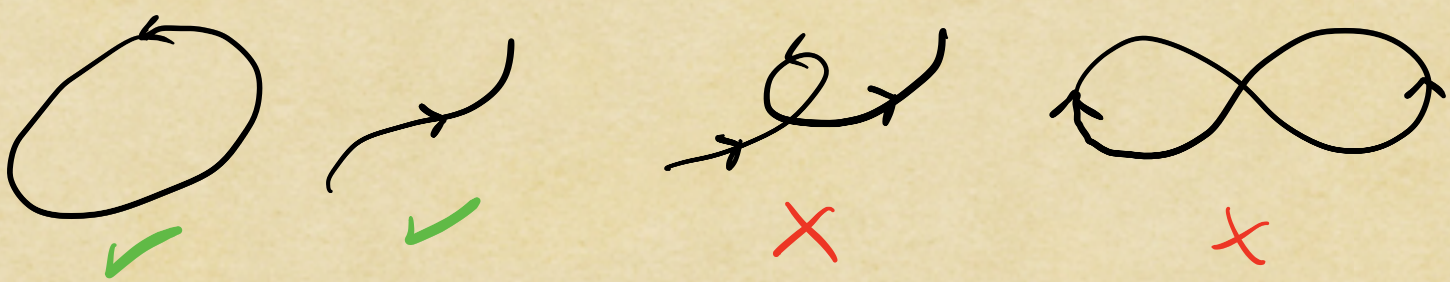
A regular curve $\vec{\beta}(t)$ is called simple if either

(1) $\vec{\beta}$ is an injective function;

(2) there's a period $T > 0$ such that

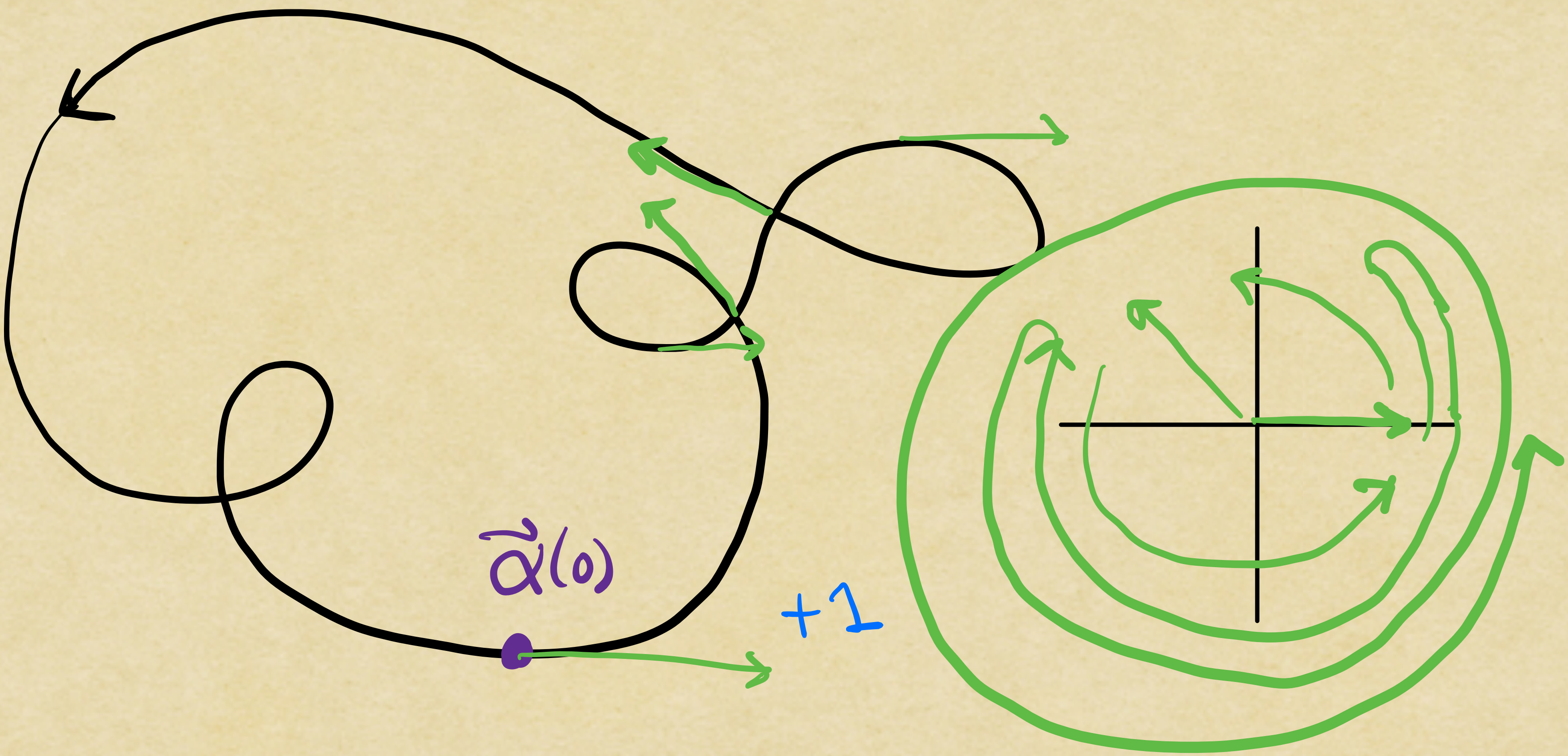
$$\vec{\beta}(t) = \vec{\beta}(\tilde{t}) \quad \text{iff} \quad t - \tilde{t} = n \cdot T,$$

for some $n \in \mathbb{Z}$.



The rotation index

Given a closed, unit-speed curve $\vec{\alpha}(s)$, we want to know the total rotation along $\vec{\alpha}$.



Intuition

- Up to reparametrization, we can choose $\alpha(0)$ so that $\vec{t}(0) = \underline{\pm \vec{e}_1}$.
- We want a function $\theta(s)$ which gives <u>the angle from $\vec{e}_1$ to $\vec{t}(s)$</u>. The natural thing to try is $\vec{t}(s) = \begin{pmatrix} \cos\theta(s) \\ \sin\theta(s) \end{pmatrix} = \begin{pmatrix} x'(s) \\ y'(s) \end{pmatrix}$

  $\longrightarrow$ take <u>inverse cosine of $x'(s)$</u>
- Problem: This will give jump discontinuities
- Fix: Define $\theta(s) := \theta_0 + \int_0^s \psi(t)\,dt$, for some carefully-chosen ψ.

How should we define θ'? $= 4$

We can't use $\vec{t}(s) = \begin{pmatrix} \cos\theta(s) \\ \sin\theta(s) \end{pmatrix}$ to define θ

directly, but we can use it to define θ'. We know that $\vec{t}(s) = \begin{pmatrix} x'(s) \\ y'(s) \end{pmatrix}$, so

$$x'(s) = \cos\theta(s) \quad ; \quad y'(s) = \sin\theta(s)$$

Differentiating,

$$x''(s) = -\theta'(s)\cdot\sin\theta(s) \quad ; \quad y''(s) = \theta'(s)\cdot\cos\theta(s)$$

In matrix form:

$$\begin{pmatrix} \cos\theta(s) & \sin\theta(s) \\ -\theta'(s)\sin\theta(s) & \theta'(s)\cos\theta(s) \end{pmatrix} = \begin{pmatrix} x'(s) & y'(s) \\ x''(s) & y''(s) \end{pmatrix}.$$

Then, taking determinants,

$$\theta'(s) = x'(s)\,y''(s) - x''(s)\,y'(s)$$

So it makes sense to declare that

$$\boxed{\theta'(s) := \underbrace{x'(s)\,y''(s) - x''(s)\,y'(s)}_{\text{planar curvature}}}$$

More carefully:

Lemma. Let $\vec{u}(s) = (f(s), g(s))$ be a unit vector, for all $s \in (a,b)$. The function

$$\theta(s) := \theta_0 + \int_{s_0}^{s} \left(f(t) \cdot g'(t) - f'(t) g(t)\right) dt$$

is smooth, where $s_0 \in (a,b)$ is fixed and $\theta_0 \in \mathbb{R}$ is any value s.t. $\vec{u}(0) = (\cos\theta_0, \sin\theta_0)$. Moreover, $\vec{u}(s) = \left(\cos\theta(s), \sin\theta(s)\right)$, $\forall s \in (a,b)$.

(Proof.) We'll just check that

$$\left(f(s), g(s)\right) = \left(\cos\theta(s), \sin\theta(s)\right)$$

Start with some observations:

1. $f^2+g^2 = 1 \Rightarrow 2f\cdot f' + 2\cdot g\cdot g' = 0$

   $\therefore ff' + gg' = 0$

2. By the FTC, $\theta'(s) = fg' - f'g$

3. Our vector equality holds iff the function

$$E(s) := \left(f(s) - \cos\theta(s)\right)^2 + \left(g(s) - \sin\theta(s)\right)^2$$

identically zero

$$E(s) = (f - \cos\theta)^2 + (g - \sin\theta)^2$$

$$\Rightarrow E'(s) = 2(f - \cos\theta)(f' + \theta' \cdot \sin\theta) + 2(g - \sin\theta)(g' - \theta' \cdot \cos\theta)$$

$$= 2\left[\cancel{ff'} - f' \cdot \cos\theta + f \cdot \theta' \cdot \sin\theta \cancel{- \theta' \cdot \cos\theta \sin\theta}\right]$$

$$+ 2\left[\cancel{gg'} - g' \cdot \sin\theta - g \cdot \theta' \cdot \cos\theta \cancel{+ \theta' \cdot \cos\theta \sin\theta}\right]$$

$$= 2\sin\theta\left[f \cdot \theta' - g'\right] - 2\cos\theta\left[g \cdot \theta' + f'\right] \qquad 1 - g^2 = f^2$$

$$f^2 - 1 = -g^2 \qquad = 2\sin\theta\left[f^2 g' - f \cdot f' g - g'\right] - 2\cos\theta\left[f g g' - f' \cdot g^2 + f'\right]$$

$$= 2\sin\theta\left[(-g^2) \cdot g' - f \cdot f' \cdot g\right] - 2\cos\theta\left[f g g' + f' \cdot f^2\right]$$

$$= -2g\sin\theta\left[gg' + ff'\right] - 2f\cos\theta\left[gg' + ff'\right]$$

$$= 0.$$

So our error function $E(s)$ is Constant

Since $E(s_0) = 0$, we win. $\diamond$

What's the point? We can use $\theta(s)$ to measure the total rotation of $\vec{u}(s)$ over its domain.

Just tracking the angle from $\vec{e}_1$ to $\vec{t}(s)$ led to jump discontinuities.

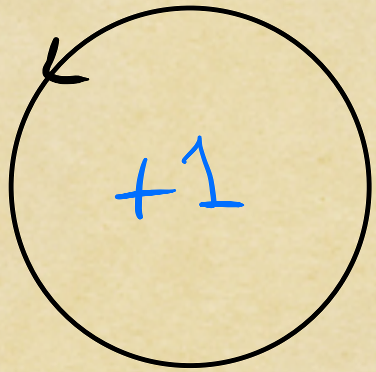
Back to our unit-speed curve $\vec{\alpha}(s)$. Applying the lemma to $\vec{t}(s) = (x'(s), y'(s))$ gives

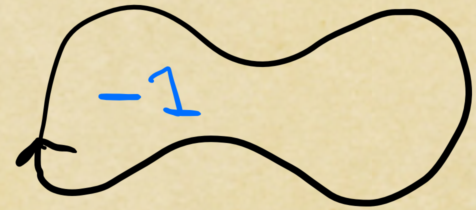
$$\theta(s) = \theta_0 + \int_0^s \left(x'(t)\, y''(t) - x''(t)\, y'(t) \right) dt$$

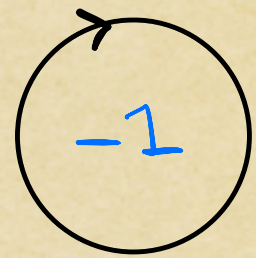
With $\theta(s)$ defined, we can finally define:

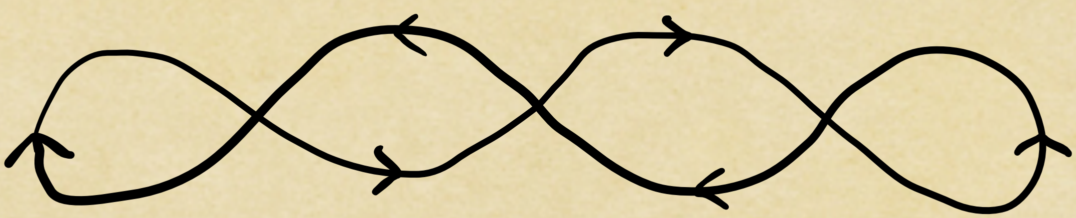
> The rotation index of a closed, unit-speed curve $\vec{\alpha}(s)$ is defined to be
>
> $$i_{\vec{\alpha}} := \frac{\theta(L) - \theta(0)}{2\pi},$$
>
> where L is the perimeter of $\vec{\alpha}$.

Examples









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