Math 4441 Practice Midterm 2

November 20, 2022

Name: ______________________ **gtID**: ______________________

Instructions. Read each question carefully and show all your work. Answers without justification will receive little to no credit. Writing your answers in a legible, well-organized manner will maximize your opportunities for partial credit.

This is a closed-note, closed-book exam, and you are expected to abide by the Georgia Tech Honor Challenge. Good luck!

Clearly label any extra papers you want graded.

By signing below, I certify that all work submitted on this exam is my own, and that I have neither given nor received any unauthorized help on this exam.

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Question	Points	Score
1	10	
2	10	
3	10	
4	10	
Total:	40	

Formulas

- $df_p = \begin{pmatrix} \frac{\partial f^1}{\partial u^1}(p) & \cdots & \frac{\partial f^1}{\partial u^m}(p) \\ \vdots & \ddots & \vdots \\ \frac{\partial f^n}{\partial u^1}(p) & \cdots & \frac{\partial f^n}{\partial u^m}(p) \end{pmatrix}$
- $d(g \circ f)_p = dg_{f(p)} \cdot df_p$
- $\frac{\partial (g\circ f)^i}{\partial u^j}|_p = \sum_{k=1}^m \frac{\partial g^i}{\partial v^k}|_{f(p)} \frac{\partial f^k}{\partial u^j}|_p$
- $\vec{x}_i := \frac{\partial \vec{x}}{\partial u^i}$
- $\tilde{\vec{x}} = \vec{x} \circ F$ implies $\tilde{\vec{x}}_j = \sum_{i=1}^2 \frac{\partial F^i}{\partial \tilde{u}^j} \vec{x}_j$
- $\vec{n} := \frac{\vec{x}_1 \times \vec{x}_2}{\|\vec{x}_1 \times \vec{x}_2\|}$
- $I_p(\vec{X}, \vec{Y}) := \langle \vec{X}, \vec{Y} \rangle_{\mathbb{R}^3}$
- $g_{ij} = \langle \vec{x}_i, \vec{x}_j \rangle$
- $I_p(\vec{X}, \vec{Y}) = \begin{pmatrix} X^1 \\ X^2 \end{pmatrix}^T \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \begin{pmatrix} Y^1 \\ Y^2 \end{pmatrix}$
- $\|\vec{X}\| := \sqrt{I_p(\vec{X}, \vec{X})}$ and $\measuredangle(\vec{X}, \vec{Y}) := \arccos\left(\frac{I_p(\vec{X},\vec{Y})}{\|\vec{X}\| \cdot \|\vec{Y}\|}\right)$
- $(g^{k\ell}) := (g_{ij})^{-1}$
- $\tilde{\vec{x}} = \vec{x} \circ F$ implies $\tilde{g}_{\alpha\beta} = \sum_{i,j=1}^2 \frac{\partial F^i}{\partial \tilde{u}^\alpha} \frac{\partial F^j}{\partial \tilde{u}^\beta} g_{ij}$
- $\vec{x}_{ij} := \frac{\partial^2 \vec{x}}{\partial u^i \partial u^j}$
- $\kappa_n := \langle \frac{d}{ds}\vec{T}, \vec{n} \rangle$ and $\kappa_g := \langle \frac{d}{ds}\vec{T}, \vec{S} \rangle$
- $L_{ij} := \langle \vec{x}_{ij}, \vec{n} \rangle$
- $\Gamma_{ij}^k := \sum_{\ell=1}^2 \langle \vec{x}_{ij}, \vec{x}_\ell \rangle g^{k\ell}$
- For $\vec{\alpha}$ unit-speed, $\vec{\alpha}'' = \kappa_n \vec{n} + \kappa_g \vec{S}$
- $\kappa_n = \sum_{i,j=1}^2 (\alpha_U^i)'(\alpha_U^j)' L_{ij}$ and $\kappa_g \vec{S} = \sum_{k=1}^2 \left((\alpha_U^k)'' + \sum_{i,j=1}^2 (\alpha_U^i)'(\alpha_U^j)'\Gamma_{ij}^k\right) \vec{x}_k$
- $D_{\vec{v}} f := (f \circ \vec{\alpha})'(0)$, where $\vec{\alpha}(0) = p$ and $\vec{\alpha}'(0) = \vec{v}$
- $\vec{n} = \nu \circ \vec{x}$
- $\mathcal{L}(\vec{v}) := -D_{\vec{v}}\nu$
- $\vec{n}_i := \frac{\partial \vec{n}}{\partial u^i}$
- $\vec{n}_j = -\sum_{i=1}^2 L_j^i \, \vec{x}_i$
- $(L_{ij}) = (g_{ij})(L_j^i)$, so $(L_j^i) = (g^{k\ell})(L_{ij})$

1. Let $f\colon U \to \mathbb{R}$ be a C^∞ function on some open set $U \subset \mathbb{R}^2$. The **graph of** f is the image of the function $\vec{x}\colon U \to \mathbb{R}^3$ defined by

$$\vec{x}(u^1, u^2) := (u^1, u^2, f(u^1, u^2)).$$

(a) (6 points) Verify (using the definition) that $\vec{x}$ is a simple surface.
Hint: You'll need to compute $\vec{x}_1 \times \vec{x}_2$.

Solution: We need to prove that $\vec{x}$ is injective and C^k for some k, and also that $\vec{x}$ is regular, in the sense that $\vec{x}_1$ and $\vec{x}_2$ are linearly independent. In fact, $\vec{x}$ is C^∞, since f is C^∞. Injectivity is also pretty straightforward: if $\vec{x}(a,b) = \vec{x}(c,d)$, then

$$(a, b, f(a,b)) = (c, d, f(c,d)) \quad \Rightarrow \quad a = c \text{ and } b = d.$$

The only part that requires some care is regularity. As indicated by the hint, we'll verify that $\vec{x}_1 \times \vec{x}_2$ is nonzero. We have

$$\vec{x}_1 = (1, 0, f_1) \quad \text{and} \quad \vec{x}_2 = (0, 1, f_2),$$

where $f_i := \frac{\partial f}{\partial u^i}$. Then

$$\vec{x}_1 \times \vec{x}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & f_1 \\ 0 & 1 & f_2 \end{vmatrix} = (-f_1, -f_2, 1).$$

Since the third component of this cross product never vanishes, the vector itself cannot vanish. So $\vec{x}$ is a simple surface.

(b) (4 points) Give formulas for the matrices (g_{ij}) and $(g^{k\ell})$ in terms of the function f. You should use the notation $f_i := \frac{\partial f}{\partial u^i}$ for partial derivatives of f, and should simplify your answers.

Solution: We've already computed $\vec{x}_1$ and $\vec{x}_2$, so we can get (g_{ij}) by taking inner products. We have

$$\langle \vec{x}_1, \vec{x}_1 \rangle = 1 + f_1^2, \quad \langle \vec{x}_1, \vec{x}_2 \rangle = f_1 f_2, \quad \text{and} \quad \langle \vec{x}_2, \vec{x}_2 \rangle = 1 + f_2^2,$$

so

$$\boxed{(g_{ij}) = \begin{pmatrix} 1 + f_1^2 & f_1 f_2 \\ f_1 f_2 & 1 + f_2^2 \end{pmatrix}}.$$

We get $(g^{k\ell})$ by inverting (g_{ij}), and this can be done with the usual 2×2 formula:

$$(g^{k\ell}) = (g_{ij})^{-1} = \frac{1}{\det(g_{ij})} \begin{pmatrix} g_{22} & -g_{12} \\ -g_{12} & g_{11} \end{pmatrix} = \boxed{\frac{1}{1 + f_1^2 + f_2^2} \begin{pmatrix} 1 + f_2^2 & -f_1 f_2 \\ -f_1 f_2 & 1 + f_1^2 \end{pmatrix}}.$$

2. Let $\vec{\alpha}\colon (a,b) \to \mathbb{R}^2$ be a unit-speed planar curve. For each $t \in \mathbb{R}$, define a planar curve $\vec{\alpha}_t\colon (a,b) \to \mathbb{R}^2$ by

$$\vec{\alpha}_t(s) := \vec{\alpha}(s) + t\,k(s)\,\vec{n}(s),$$

where $\vec{n}(s) = J\vec{T}(s)$ is the planar normal vector* for $\vec{\alpha}$ and $k(s)$ is the planar curvature. Note that $\vec{\alpha}_t$ is not necessarily unit-speed. Finally, let $L\colon \mathbb{R} \to [0,\infty)$ be defined by

$$L(t) := \ell(\vec{\alpha}_t) = \int_a^b \|\vec{\alpha}_t'(s)\|\,ds.$$

Note: This problem is way too long to actually show up on a midterm. But you should expect some much smaller version of this type of thing, where you need to know how a technical computation goes.

(a) (3 points) Prove that

$$L'(t) = \int_a^b \frac{\left\langle \frac{\partial^2}{\partial s \partial t}\vec{\alpha}_t, \frac{\partial}{\partial s}\vec{\alpha}_t \right\rangle}{\sqrt{\left\langle \frac{\partial}{\partial s}\vec{\alpha}_t, \frac{\partial}{\partial s}\vec{\alpha}_t \right\rangle}}\,ds.$$

Solution: Because the integral that occurs in the definition of $L(t)$ is taken with respect to s, the t-derivative can slide right past:

$$L'(t) = \frac{\partial}{\partial t}\left(\int_a^b \sqrt{\langle \vec{\alpha}_t'(s), \vec{\alpha}_t'(s)\rangle}\,ds\right) = \int_a^b \frac{\partial}{\partial t}\left(\sqrt{\langle \vec{\alpha}_t'(s), \vec{\alpha}_t'(s)\rangle}\right)ds$$
$$= \int_a^b \frac{\frac{\partial}{\partial t}\left(\langle \vec{\alpha}_t'(s), \vec{\alpha}_t'(s)\rangle\right)}{2\sqrt{\langle \vec{\alpha}_t'(s), \vec{\alpha}_t'(s)\rangle}}\,ds = \int_a^b \frac{\frac{\partial}{\partial t}\left(\langle \frac{\partial}{\partial s}\vec{\alpha}_t, \frac{\partial}{\partial s}\vec{\alpha}_t\rangle\right)}{2\sqrt{\langle \frac{\partial}{\partial s}\vec{\alpha}_t, \frac{\partial}{\partial s}\vec{\alpha}_t\rangle}}\,ds.$$

The penultimate equality above uses the chain rule, while the last one declutters notation a bit. We obtain the desired quantity by applying the product rule:

$$L'(t) = \int_a^b \frac{\frac{\partial}{\partial t}\left(\langle \frac{\partial}{\partial s}\vec{\alpha}_t, \frac{\partial}{\partial s}\vec{\alpha}_t(s)\rangle\right)}{2\sqrt{\langle \frac{\partial}{\partial s}\vec{\alpha}_t, \frac{\partial}{\partial s}\vec{\alpha}_t\rangle}}\,ds = \int_a^b \frac{2\langle \frac{\partial^2}{\partial t \partial s}\vec{\alpha}_t, \frac{\partial}{\partial s}\vec{\alpha}_t\rangle}{2\sqrt{\langle \frac{\partial}{\partial s}\vec{\alpha}_t, \frac{\partial}{\partial s}\vec{\alpha}_t\rangle}}\,ds$$

A quick cancellation (and equality of mixed partial derivatives) yields the desired integral.

(b) (2 points) Given the formula in part (a), explain why

$$L'(0) = \int_a^b \left\langle \frac{\partial^2}{\partial s \partial t}\vec{\alpha}_t, \frac{\partial}{\partial s}\vec{\alpha}_t \right\rangle\Bigg|_{t=0}\,ds.$$

Solution: We can compute $L'(0)$ by evaluating the formula in part (a) at $t = 0$. In the denominator this produces

$$\sqrt{\left\langle \frac{\partial}{\partial s}\vec{\alpha}_0, \frac{\partial}{\partial s}\vec{\alpha}_0 \right\rangle} = \sqrt{\left\langle \frac{\partial}{\partial s}\vec{\alpha}, \frac{\partial}{\partial s}\vec{\alpha} \right\rangle} = 1,$$

since $\vec{\alpha}$ is a unit-speed curve. Thus the expression simplifies to just the numerator, evaluated at $t = 0$. This gives the equation we're asked to justify.

*In class we just wrote $J\vec{T}$ over and over, but here we're calling it $\vec{n}$.

(c) (3 points) Use integration by parts to show that in fact

$$L'(0) = -\int_a^b \left\langle \frac{\partial}{\partial t}\vec{\alpha}_t, \frac{\partial^2}{\partial s^2}\vec{\alpha}_t \right\rangle\Big|_{t=0} ds.$$

Solution: Integration by parts undoes the product rule, so let's find a product whose derivative will be related to the quantity we're considering. We have

$$\frac{\partial}{\partial s}\left\langle \frac{\partial}{\partial t}\vec{\alpha}_t, \frac{\partial}{\partial s}\vec{\alpha}_t \right\rangle = \left\langle \frac{\partial^2}{\partial s \partial t}\vec{\alpha}_t, \frac{\partial}{\partial s}\vec{\alpha}_t \right\rangle + \left\langle \frac{\partial}{\partial t}\vec{\alpha}_t, \frac{\partial^2}{\partial s^2}\vec{\alpha}_t \right\rangle,$$

and thus

$$\left\langle \frac{\partial^2}{\partial s \partial t}\vec{\alpha}_t, \frac{\partial}{\partial s}\vec{\alpha}_t \right\rangle = \frac{\partial}{\partial s}\left\langle \frac{\partial}{\partial t}\vec{\alpha}_t, \frac{\partial}{\partial s}\vec{\alpha}_t \right\rangle - \left\langle \frac{\partial}{\partial t}\vec{\alpha}_t, \frac{\partial^2}{\partial s^2}\vec{\alpha}_t \right\rangle.$$

Now, since $\vec{\alpha}_t = \vec{\alpha} + t\,k\,\vec{n}$, $\frac{\partial}{\partial t}\vec{\alpha}_t = k\,\vec{n}$. This is perpendciular to $\vec{\alpha}_t' = \frac{\partial}{\partial s}\vec{\alpha}_t$, and thus the first term on the right hand side above vanishes. So we have

$$L'(0) = \int_a^b \left\langle \frac{\partial^2}{\partial s \partial t}\vec{\alpha}_t, \frac{\partial}{\partial s}\vec{\alpha}_t \right\rangle\Big|_{t=0} ds = -\int_a^b \left\langle \frac{\partial}{\partial t}\vec{\alpha}_t, \frac{\partial^2}{\partial s^2}\vec{\alpha}_t \right\rangle\Big|_{t=0} ds,$$

as desired.

(d) (2 points) By plugging into the formula just derived, show that

$$L'(0) = \int_a^b -k(s)^2\, ds.$$

Solution: We previously showed that $\frac{\partial}{\partial t}\vec{\alpha}_t = k\,\vec{n}$, so we have

$$L'(0) = -\int_a^b \left\langle \frac{\partial}{\partial t}\vec{\alpha}_t, \frac{\partial^2}{\partial s^2}\vec{\alpha}_t \right\rangle\Big|_{t=0} ds = -\int_a^b \langle k\,\vec{n}, \vec{\alpha}_t''\rangle|_{t=0}\, ds = -\int_a^b k\langle \vec{n}, \vec{\alpha}''\rangle\, ds,$$

where the last equality uses the fact that $\vec{\alpha}_0 = \vec{\alpha}$. Finally, we recognize $\langle \vec{n}, \vec{\alpha}''\rangle$ as giving the planar curvature k, and thus

$$L'(0) = -\int_a^b (k(s))^2\, ds,$$

as desired.

This part wasn't asked of you, but remember why we care: because $\vec{\alpha}$ is length-minimizing, we know that $L'(0) = 0$; this computation implies that $k(s) = 0$, for all $s \in (a, b)$.

3. Consider the simple surface $\vec{x}\colon (-\pi,\pi)\times\mathbb{R}\to\mathbb{R}^3$ defined by

$$\vec{x}(u^1,u^2) := (\cos u^1, \sin u^1, u^1+u^2),$$

whose image is the cylinder $x^2+y^2=1$.

(a) (4 points) Compute the coefficients L_{ij} of the second fundamental form. Recall that $L_{ij} := \langle \vec{x}_{ij}, \vec{n}\rangle$.

Solution: We start by computing the first derivatives:

$$\vec{x}_1 = (-\sin u^1, \cos u^1, 1) \quad \text{and} \quad \vec{x}_2 = (0,0,1).$$

From these we compute $\vec{n}$, starting with the cross product:

$$\vec{x}_1\times\vec{x}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\sin u^1 & \cos u^1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = (\cos u^1, \sin u^1, 0).$$

Since this vector has unit length, we find that $\vec{n} = (\vec{x}_1\times\vec{x}_2)/|\vec{x}_1\times\vec{x}_2| = \vec{x}_1\times\vec{x}_2$. Next, the second derivatives:

$$\vec{x}_{11} = (-\cos u^1, -\sin u^1, 0), \quad \vec{x}_{12} = (0,0,0), \quad \text{and} \quad \vec{x}_{22} = (0,0,0).$$

Thus we immediately have $\boxed{L_{12}=0}$ and $\boxed{L_{22}=0}$, and quickly find that

$$\boxed{L_{11} = \langle \vec{x}_{11}, \vec{n}\rangle = \langle(-\cos u^1, -\sin u^1, 0), (\cos u^1, \sin u^1, 0)\rangle = -1}.$$

In matrix form:

$$\boxed{(L_{ij}) = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}.}$$

(You were not required to put this in matrix form.)

(b) (6 points) Use the formula $(L^i_j) = (g_{ij})^{-1}(L_{ij})$ to compute the matrix representation (L^i_j) of the Weingarten map.
Note: You will need to compute (g_{ij}) along the way.

Solution: From the above computation of $\vec{x}_1$ and $\vec{x}_2$ we can find that

$$(g_{ij}) = \begin{pmatrix} \langle\vec{x}_1,\vec{x}_1\rangle & \langle\vec{x}_1,\vec{x}_2\rangle \\ \langle\vec{x}_1,\vec{x}_2\rangle & \langle\vec{x}_2,\vec{x}_2\rangle \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

Then

$$(L^i_j) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} = \boxed{\begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix}}.$$

(c) (*Bonus, but not really because there aren't any bonus points, and also this is a practice midterm*) The matrices $(g_{ij})^{-1}$ and (L_{ij}) are symmetric; how does it happen that (L^i_j) is not symmetric?

Solution: The short answer is that this is how transposes work. A product of symmetric matrices need not be symmetric, if the matrices don't commute: $(AB)^T = B^TA^T = BA$, if A and B are symmetric, but this doesn't have to equal AB.

4. (a) (3 points) Let $\{\vec{T}(s), \vec{S}(s), \vec{n}(s)\}$ be the Darboux frame of a unit-speed surface curve. Prove that $\frac{d}{ds}\vec{T}(s)$ lies in the plane spanned by $\vec{S}(s)$ and $\vec{n}(s)$.

Solution: Because the Darboux frame gives an orthonormal basis, $\vec{S}$ and $\vec{n}$ form a basis for the plane perpendicular to $\vec{T}$. At the same time, $\langle \vec{T}, \vec{T} \rangle \equiv 1$, so we can take the s-derivative and use the chain rule to find that $2\langle \frac{d}{ds}\vec{T}, \vec{T} \rangle \equiv 0$. So $\frac{d}{ds}\vec{T}$ is everywhere perpendicular to $\vec{T}$, and thus lies in the plane spanned by $\vec{S}$ and $\vec{n}$.

(b) (2 points) The following equation† tells us why we care about the Christoffel symbols Γ_{ij}^k:

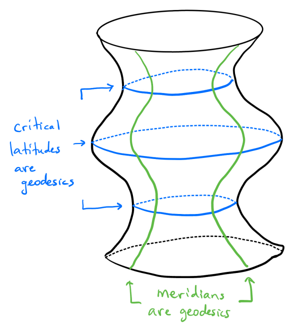
$$\vec{x}_{ij} = L_{ij}\,\vec{n} + \sum_{k=1}^{2} \Gamma_{ij}^k\,\vec{x}_k.$$

Interpret this equation by writing a sentence that begins with, "The Christoffel symbols tell us..."

Solution: The Christoffel symbols tell us **the tangential components of the second derivatives $\vec{x}_ij$.**

(c) (3 points) Sketch a surface of revolution for which exactly three latitudes are geodesics. Identify these geodesics, and sketch at least two additional geodesics.

Solution:

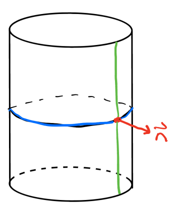


(d) (2 points) Sketch a surface $\mathcal{S}$ and choose a point $p \in \mathcal{S}$ such that the Weingarten map can be represented by the matrix

$$\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Indicate the point p that you've chosen, and **draw the surface normal $\vec{n}$ at this point**. You don't have to justify your sketch, but you are welcome to add words to explain what you've drawn.

Solution:



†Notice that this equation is *not* on your formula sheet. It also won't be on the formula sheet for the real midterm.