Math 4441 Practice Midterm 1
Fall 2022

Name: ______________________ **gtID**: ______________________

Instructions. Read each question carefully and show all your work. Answers without justification will receive little to no credit. Writing your answers in a legible, well-organized manner will maximize your opportunities for partial credit.

This is a closed-note, closed-book exam, and you are expected to abide by the Georgia Tech Honor Challenge. Good luck!

Clearly label any extra papers you want graded.

By signing below, I certify that all work submitted on this exam is my own, and that I have neither given nor received any unauthorized help on this exam.

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Question	Points	Score
1	10	
2	10	
3	10	
4	10	
Total:	40	

Formulas

- Tangent line: $\vec{\ell}(\lambda) := \vec{\alpha}(t_0) + \lambda \vec{T}(t_0)$
- Arc length: $\int_a^b \|\vec{\alpha}(t)\|\, dt$
- Magnitude: $\|\vec{v}\| := \sqrt{\vec{v} \cdot \vec{v}}$
- Angle: $\measuredangle(\vec{v}, \vec{w}) := \cos^{-1}\left(\frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\| \cdot \|\vec{w}\|}\right)$
- Cauchy-Schwarz: $|\langle \vec{v}, \vec{w} \rangle| \leq \|\vec{v}\| \cdot \|\vec{w}\|$, with equality iff $\vec{v} \parallel \vec{w}$.
- Curvature: $\kappa(s) := \|\frac{d}{ds}\vec{T}(s)\|$
- For $\vec{\alpha}$ unit-speed, with $\kappa(s) \neq 0$, $\vec{B} := \vec{T} \times \vec{N}$.
- Torsion: $\tau(s) := -\langle \vec{B}'(s), \vec{N}(s) \rangle$
- For $\vec{\alpha}$ unit-speed, with $\kappa(s) \neq 0$:

$$\begin{pmatrix} \vec{T}'(s) \\ \vec{N}'(s) \\ \vec{B}'(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix} \begin{pmatrix} \vec{T}(s) \\ \vec{N}(s) \\ \vec{B}(s) \end{pmatrix}.$$

- For $\vec{\alpha}$ regular,

$$\kappa(t) = \frac{\|\vec{\alpha}'(t) \times \vec{\alpha}''(t)\|}{\|\vec{\alpha}'(t)\|^3} \quad \text{and} \quad \tau(t) = \frac{\langle \vec{\alpha}'(t) \times \vec{\alpha}''(t), \vec{\alpha}'''(t) \rangle}{\|\vec{\alpha}'(t) \times \vec{\alpha}''(t)\|^2}.$$

- For $\vec{\alpha}$ planar and unit-speed, $k(s) := \langle \frac{d}{ds}\vec{t}(s), \vec{n}(s) \rangle$.
- For $\vec{\alpha}$ planar and regular,

$$k(t) = \frac{\langle \vec{\alpha}''(t), J(\vec{\alpha}'(t)) \rangle}{\|\vec{\alpha}'(t)\|^3} = \frac{x'(t)y''(t) - x''(t)y'(t)}{((x'(t))^2 + (y'(t))^2)^{3/2}}.$$

- If $k(t) \neq 0$, $\vec{\varepsilon}(t) = \vec{\alpha}(t) + \frac{1}{k(t)}\vec{n}(t)$.
- $\int_{\vec{\alpha}} f\, dx + g\, dy := \int_a^b \langle f(\vec{\alpha}(t)), g(\vec{\alpha}(t)) \rangle \cdot \vec{\alpha}'(t)\, dt$
- If $\vec{\alpha}$ is regular, bounds a region $\mathcal{R} \subset \mathbb{R}^2$, and is oriented counter-clockwise, then

$$\oint_{\vec{\alpha}} f\, dx + g\, dy = \iint_{\mathcal{R}} \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA.$$

- Angular rotation function: $\theta(s) := \theta_0 + \int_0^s k(u)\, du$.
- Rotation index: $i_{\vec{\alpha}} := \dfrac{\theta(L) - \theta(0)}{2\pi}$.

1. Let $\vec{\alpha}(t) = (r\cos t, r\sin t, t)$, for $t \in (-\infty, \infty)$, where $r > 0$ is a constant.
   (a) (3 points) Verify that $\vec{\alpha}$ is regular.
   (b) (3 points) Reparametrize $\vec{\alpha}$ by arc length.
   (c) (4 points) Show that the curvature κ is bounded above by $1/2$, regardless of the value of $r > 0$.
   Hint: Use your arclength parametrization to compute κ, and then use the fact that $0 \leq (r-1)^2$.

2. (10 points) Let $\vec{\alpha}(s)$ be an arbitrary unit-speed curve in $\mathbb{R}^3$ with nonvanishing curvature $\kappa(s) > 0$. Prove that

$$\vec{N}'(s) = -\kappa(s)\,\vec{T}(s) + \tau(s)\,\vec{B}(s).$$

Note: Don't just cite something — recreate the proof from class.

3. Let $\vec{\alpha}\colon [0, L] \to \mathbb{R}^2$ be a simple, closed, planar, unit-speed curve. Assume that $\vec{\alpha}$ is oriented counter-clockwise. For any constant $r \in \mathbb{R}$, consider the curve

$$\vec{\beta}(t) := \vec{\alpha}(t) - r\,\vec{n}(t).$$

   (a) (3 points) Show that $\vec{\beta}$ is regular if and only if the planar curvature of $\vec{\alpha}$ is nowhere equal to $-1/r$.

   (b) (3 points) Assuming $1 + r\,k(t) > 0$ for all t, show that* $\text{length}(\vec{\beta}) = \text{length}(\vec{\alpha}) + 2\pi\,r$.

   (c) (4 points) Show that the planar curvatures of $\vec{\alpha}$ and $\vec{\beta}$ satisfy $k_{\vec{\beta}}(t) = k_{\vec{\alpha}}(t)/(1 + r\,k_{\vec{\alpha}}(t))$.

*Here's a cute example of this equation: consider two strings, one which is wrapped around a tennis ball and one which is wrapped around Earth's equator. The amount of additional length needed in order to pull each of these strings one inch off of their respective surfaces is the same — 2π inches.

4. (a) (3 points) Give an example of an isometry $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ with $f(\vec{0}) \neq \vec{0}$ and $f(\vec{e}_1) \neq \vec{e}_1 + f(\vec{0})$.
Note: Give an actual formula, not just a description in words.

   (b) (2 points) Identify two reasons that there does not exist a unit-speed curve $\vec{\alpha}\colon \mathbb{R} \to \mathbb{R}^3$ whose Frenet frame satisfies the equations

$$\begin{pmatrix} \vec{T}' \\ \vec{N}' \\ \vec{B}' \end{pmatrix} = \begin{pmatrix} 0 & s^3 & 0 \\ -s^3 & 0 & \cos s \\ 0 & \cos s & 0 \end{pmatrix} \begin{pmatrix} \vec{T} \\ \vec{N} \\ \vec{B} \end{pmatrix}.$$

   (c) (2 points) Draw an oriented, closed, planar curve $\vec{\alpha}$ with the property that $\int_{\vec{\alpha}} k(s)\,ds = -6\pi$, where $k(s)$ is the planar curvature. No justification needed.

   (d) (3 points) Is there a simple, closed curve in the plane with length equal to 6 meters and bounding an area of 3 square meters? Justify your answer.