Math 4441 Practice Midterm 1

Fall 2022

Name: ______________________ **gtID**: ______________________

Instructions. Read each question carefully and show all your work. Answers without justification will receive little to no credit. Writing your answers in a legible, well-organized manner will maximize your opportunities for partial credit.

This is a closed-note, closed-book exam, and you are expected to abide by the Georgia Tech Honor Challenge. Good luck!

Clearly label any extra papers you want graded.

By signing below, I certify that all work submitted on this exam is my own, and that I have neither given nor received any unauthorized help on this exam.

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Question	Points	Score
1	10	
2	10	
3	10	
4	10	
Total:	40	

Formulas

- Tangent line: $\vec{\ell}(\lambda) := \vec{\alpha}(t_0) + \lambda\,\vec{T}(t_0)$
- Arc length: $\int_a^b \|\vec{\alpha}(t)\|\,dt$
- Magnitude: $\|\vec{v}\| := \sqrt{\vec{v}\cdot\vec{v}}$
- Angle: $\measuredangle(\vec{v},\vec{w}) := \cos^{-1}\left(\frac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\cdot\|\vec{w}\|}\right)$
- Cauchy-Schwarz: $|\langle\vec{v},\vec{w}\rangle| \le \|\vec{v}\|\cdot\|\vec{w}\|$, with equality iff $\vec{v}\parallel\vec{w}$.
- Curvature: $\kappa(s) := \|\frac{d}{ds}\vec{T}(s)\|$
- For $\vec{\alpha}$ unit-speed, with $\kappa(s)\neq 0$, $\vec{B} := \vec{T}\times\vec{N}$.
- Torsion: $\tau(s) := -\langle\vec{B}'(s),\vec{N}(s)\rangle$
- For $\vec{\alpha}$ unit-speed, with $\kappa(s)\neq 0$:

$$\begin{pmatrix}\vec{T}'(s)\\ \vec{N}'(s)\\ \vec{B}'(s)\end{pmatrix} = \begin{pmatrix}0 & \kappa(s) & 0\\ -\kappa(s) & 0 & \tau(s)\\ 0 & -\tau(s) & 0\end{pmatrix}\begin{pmatrix}\vec{T}(s)\\ \vec{N}(s)\\ \vec{B}(s)\end{pmatrix}.$$

- For $\vec{\alpha}$ regular,

$$\kappa(t) = \frac{\|\vec{\alpha}'(t)\times\vec{\alpha}''(t)\|}{\|\vec{\alpha}'(t)\|^3} \quad\text{and}\quad \tau(t) = \frac{\langle\vec{\alpha}'(t)\times\vec{\alpha}''(t),\vec{\alpha}'''(t)\rangle}{\|\vec{\alpha}'(t)\times\vec{\alpha}''(t)\|^2}.$$

- For $\vec{\alpha}$ planar and unit-speed, $k(s) := \langle\frac{d}{ds}\vec{t}(s),\vec{n}(s)\rangle$.
- For $\vec{\alpha}$ planar and regular,

$$k(t) = \frac{\langle\vec{\alpha}''(t), J(\vec{\alpha}'(t))\rangle}{\|\vec{\alpha}'(t)\|^3} = \frac{x'(t)y''(t) - x''(t)y'(t)}{((x'(t))^2 + (y'(t))^2)^{3/2}}.$$

- If $k(t)\neq 0$, $\vec{\varepsilon}(t) = \vec{\alpha}(t) + \frac{1}{k(t)}\vec{n}(t)$.
- $\int_{\vec{\alpha}} f\,dx + g\,dy := \int_a^b \langle f(\vec{\alpha}(t)), g(\vec{\alpha}(t))\rangle\cdot\vec{\alpha}'(t)\,dt$
- If $\vec{\alpha}$ is regular, bounds a region $\mathcal{R}\subset\mathbb{R}^2$, and is oriented counter-clockwise, then

$$\oint_{\vec{\alpha}} f\,dx + g\,dy = \iint_{\mathcal{R}}\left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)dA.$$

- Angular rotation function: $\theta(s) := \theta_0 + \int_0^s k(u)\,du$.
- Rotation index: $i_{\vec{\alpha}} := \frac{\theta(L)-\theta(0)}{2\pi}$.

1. Let $\vec{\alpha}(t) = (r\cos t, r\sin t, t)$, for $t \in (-\infty, \infty)$, where $r > 0$ is a constant.

(a) (3 points) Verify that $\vec{\alpha}$ is regular.

Solution: We have $\vec{\alpha}'(t) = (-r\sin t, r\cos t, 1)$, so

$$\|\vec{\alpha}'(t)\| = \sqrt{(-r\sin t)^2 + (r\cos t)^2 + 1} = \sqrt{r^2+1}.$$

Regardless of the value of r, this magnitude is nonzero, so $\vec{\alpha}$ is regular.

(b) (3 points) Reparametrize $\vec{\alpha}$ by arc length.

Solution: We can use our expression for $\|\vec{\alpha}'(t)\|$ to compute an arclength function:

$$h(t) = \int_0^t \|\vec{\alpha}'(u)\|\,du = \int_0^t \sqrt{r^2+1}\,du = t\sqrt{r^2+1}.$$

Notice that the range of h is $(-\infty, \infty)$. To get an arclength parametrization, we invert this. That is,

$$g(s) = h^{-1}(s) = \frac{s}{\sqrt{r^2+1}}, \quad s \in (-\infty, \infty).$$

The domain of g is determined by the range of h. Finally, we define $\vec{\beta}(s) := (\vec{\alpha} \circ g)(s)$:

$$\vec{\beta}(s) = \left(r\cos\left(\frac{s}{\sqrt{r^2+1}}\right), r\sin\left(\frac{s}{\sqrt{r^2+1}}\right), \frac{s}{\sqrt{r^2+1}}\right), \quad s \in (-\infty, \infty).$$

(c) (4 points) Show that the curvature κ is bounded above by $1/2$, regardless of the value of $r > 0$. *Hint: Use your arclength parametrization to compute κ, and then use the fact that* $0 \leq (r-1)^2$.

Solution: Since $\vec{\beta}$ is a unit-speed parametrization of our curve, $\vec{T}(s) = \vec{\beta}'(s)$, and the curvature is thus given by

$$\kappa(s) = \left\|\frac{d}{ds}\vec{T}(s)\right\| = \left\|\frac{d}{ds}\vec{\beta}'(s)\right\| = \|\vec{\beta}''(s)\|.$$

We have

$$\vec{\beta}'(s) = \left(-\frac{r}{\sqrt{r^2+1}}\sin\left(\frac{s}{\sqrt{r^2+1}}\right), \frac{r}{\sqrt{r^2+1}}\cos\left(\frac{s}{\sqrt{r^2+1}}\right), \frac{1}{\sqrt{r^2+1}}\right),$$

so

$$\vec{\beta}''(s) = \left(-\frac{r}{r^2+1}\cos\left(\frac{s}{\sqrt{r^2+1}}\right), -\frac{r}{r^2+1}\sin\left(\frac{s}{\sqrt{r^2+1}}\right), 0\right).$$

Then

$$\kappa(s) = \|\vec{\beta}''(s)\| = \frac{r}{r^2+1}.$$

Finally, because $0 \leq (r-1)^2 = r^2 - 2r + 1$, we see that

$$2r \leq r^2 + 1 \quad \Rightarrow \quad \frac{r}{r^2+1} \leq \frac{1}{2},$$

so $\kappa(s) \leq 1/2$.

2. (10 points) Let $\vec{\alpha}(s)$ be an arbitrary unit-speed curve in $\mathbb{R}^3$ with nonvanishing curvature $\kappa(s) > 0$. Prove that

$$\vec{N}'(s) = -\kappa(s)\,\vec{T}(s) + \tau(s)\,\vec{B}(s).$$

Note: Don't just cite something — recreate the proof from class.

Solution: Because $\{\vec{T}, \vec{N}, \vec{B}\}$ is an orthonormal basis for $\mathbb{R}^3$, we can write

$$\vec{v} = \langle \vec{T}, \vec{v}\rangle\,\vec{T} + \langle \vec{N}, \vec{v}\rangle\,\vec{N} + \langle \vec{B}, \vec{v}\rangle\,\vec{B},$$

for any vector $\vec{v} \in \mathbb{R}^3$. In particular,

$$\vec{N}' = \langle \vec{T}, \vec{N}'\rangle\,\vec{T} + \langle \vec{N}, \vec{N}'\rangle\,\vec{N} + \langle \vec{B}, \vec{N}'\rangle\,\vec{B}.$$

Now we know that $\langle \vec{T}, \vec{N}\rangle \equiv 0$. Differentiating (using the product rule) yields

$$\langle \vec{T}', \vec{N}\rangle + \langle \vec{T}, \vec{N}'\rangle = 0,$$

so $\langle \vec{T}, \vec{N}'\rangle = -\langle \vec{T}', \vec{N}\rangle$, and thus

$$\langle \vec{T}, \vec{N}'\rangle = -\left\langle \vec{T}', \frac{\vec{T}'}{\|\vec{T}'\|}\right\rangle = -\frac{\|\vec{T}'\|^2}{\|\vec{T}'\|} = -\|\vec{T}'\|.$$

But $\|\vec{T}'\| = \kappa$, so $\boxed{\langle \vec{T}, \vec{N}'\rangle = -\kappa}$. The first equality of the previous line uses the definition of $\vec{N}$. We compute the other coefficients similarly. For instance, $\langle \vec{N}, \vec{N}\rangle \equiv 1$, so differentiating yields

$$\langle \vec{N}', \vec{N}\rangle + \langle \vec{N}, \vec{N}'\rangle = 0 \quad \Rightarrow \quad \boxed{\langle \vec{N}, \vec{N}'\rangle = 0.}$$

Finally, $\langle \vec{B}, \vec{N}\rangle \equiv 0$, so

$$\langle \vec{B}', \vec{N}\rangle + \langle \vec{B}, \vec{N}'\rangle = 0 \quad \Rightarrow \quad \langle \vec{B}, \vec{N}'\rangle = -\langle \vec{B}', \vec{N}\rangle.$$

But recall the definition: $\tau = -\langle \vec{B}', \vec{N}\rangle$. So $\boxed{\langle \vec{B}, \vec{N}'\rangle = \tau}$. Putting together all the coefficients we've computed gives us

$$\vec{N}'(s) = -\kappa(s)\,\vec{T}(s) + \tau(s)\,\vec{B}(s),$$

as desired.

3. Let $\vec{\alpha}\colon [0, L] \to \mathbb{R}^2$ be a simple, closed, planar, unit-speed curve. Assume that $\vec{\alpha}$ is oriented counterclockwise. For any constant $r \in \mathbb{R}$, consider the curve

$$\vec{\beta}(t) := \vec{\alpha}(t) - r\,\vec{n}(t).$$

(a) (3 points) Show that $\vec{\beta}$ is regular if and only if the planar curvature of $\vec{\alpha}$ is nowhere equal to $-1/r$.

Solution: We have

$$\vec{\beta}'(t) = \vec{\alpha}'(t) - r\,\vec{n}'(t) = \vec{\alpha}'(t) + r\,k(t)\,\vec{t}(t).$$

The second equality uses one of the 2D Frenet-Serret equations: $\vec{n}' = -k\,\vec{t}$. (We're also using the fact that $\vec{\alpha}$ is unit-speed, so that these equations hold.) Now $\vec{t}(t) = \vec{\alpha}'(t)$, so in fact we see that

$$\vec{\beta}'(t) = \vec{\alpha}'(t) + r\,k(t)\,\vec{\alpha}'(t) = (1 + r\,k(t))\,\vec{\alpha}'(t),$$

and thus

$$\|\vec{\beta}'(t)\| = |1 + r\,k(t)|\,\|\vec{\alpha}'(t)\| = |1 + r\,k(t)|.$$

In order for $\vec{\beta}$ to be regular, this magnitude must be nonvanishing, which is to say $1 + r\,k(t) \neq 0$. That is, $k(t) \neq -1/r$.

(b) (3 points) Assuming $1 + r\,k(t) > 0$ for all t, show that* $\text{length}(\vec{\beta}) = \text{length}(\vec{\alpha}) + 2\pi\,r$.

Solution: We can use our expression for $\|\vec{\beta}'(t)\|$:

$$\|\vec{\beta}'(t)\| = |1 + r\,k(t)| = 1 + r\,k(t),$$

since we're assuming that $1 + r\,k(t) > 0$. Then

$$\text{length}(\vec{\beta}) = \int_0^L \|\vec{\beta}'(t)\|\,dt = \int_0^L (1 + r\,k(t))\,dt = \int_0^L 1\,dt + r\int_0^L k(t)\,dt = L + r\,2\pi.$$

The last equation uses the rotation index theorem. (Because $\vec{\alpha}$ is oriented counterclockwise, we're assured of getting 2π instead of -2π.) Since $L = \text{length}(\vec{\alpha})$, this proves the desired equation. (This equation for $\text{length}(\vec{\beta})$ is also true if $1 + r\,k(t) < 0$ for all t.)

(c) (4 points) Show that the planar curvatures of $\vec{\alpha}$ and $\vec{\beta}$ satisfy $k_{\vec{\beta}}(t) = k_{\vec{\alpha}}(t)/(1 + r\,k_{\vec{\alpha}}(t))$.

Solution: Since $\vec{\beta}$ is probably not unit-speed, we'll need to use the non-unit-speed equation for planar curvature

$$k_{\vec{\beta}}(t) = \frac{\langle \vec{\beta}''(t), J(\vec{\beta}'(t))\rangle}{\|\vec{\beta}'(t)\|^3}.$$

We have $\vec{\beta}'(t) = (1 + r\,k_{\vec{\alpha}}(t))\,\vec{\alpha}'(t)$ from above, and thus

$$\vec{\beta}''(t) = (1 + r\,k_{\vec{\alpha}}(t))\,\vec{\alpha}''(t) + r\,k'_{\vec{\alpha}}(t)\,\vec{\alpha}'(t).$$

So

$$\langle \vec{\beta}''(t), J(\vec{\beta}'(t))\rangle = \langle (1 + r\,k_{\vec{\alpha}}(t))\,\vec{\alpha}''(t) + r\,k'_{\vec{\alpha}}(t)\,\vec{\alpha}'(t), (1 + r\,k_{\vec{\alpha}}(t))J\vec{\alpha}'(t)\rangle$$

*Here's a cute example of this equation: consider two strings, one which is wrapped around a tennis ball and one which is wrapped around Earth's equator. The amount of additional length needed in order to pull each of these strings one inch off of their respective surfaces is the same — 2π inches.

$$= (1 + r\,k'_{\vec{\alpha}}(t))^2 \langle \vec{\alpha}''(t), J\vec{\alpha}'(t)\rangle = \|\vec{\beta}'(t)\|^2\,\langle \vec{\alpha}''(t), J\vec{\alpha}'(t)\rangle.$$

The penultimate equality uses the fact that $\langle \vec{\alpha}'(t), J\vec{\alpha}'(t)\rangle = 0$. Plugging into our expression for $k_{\vec{\beta}}$ yields

$$\begin{aligned} k_{\vec{\beta}}(t) &= \frac{\langle \vec{\beta}''(t), J(\vec{\beta}'(t))\rangle}{\|\vec{\beta}'(t)\|^3} \\ &= \frac{\langle \vec{\alpha}''(t), J\vec{\alpha}'(t)\rangle}{\|\vec{\beta}'(t)\|} \\ &= \frac{\langle \vec{\alpha}''(t), J\vec{\alpha}'(t)\rangle}{(1 + r\,k_{\vec{\alpha}}(t))\|\vec{\alpha}'(t)\|} \\ &= \frac{k_{\vec{\alpha}}(t)}{(1 + r\,k_{\vec{\alpha}}(t))}, \end{aligned}$$

as desired.

4. (a) (3 points) Give an example of an isometry $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ with $f(\vec{0}) \neq \vec{0}$ and $f(\vec{e}_1) \neq \vec{e}_1 + f(\vec{0})$.
Note: Give an actual formula, not just a description in words.

Solution: Remember that isometries have the form $\vec{x} \mapsto A\vec{x} + \vec{c}$, for some constant orthogonal matrix A and constant vector $\vec{c}$, so we need to choose A and $\vec{c}$ to satisfy the given conditions. We have

$$f(\vec{0}) = \vec{c} \quad \text{and} \quad f(\vec{e}_1) = A\vec{e}_1 + \vec{c} = A\vec{e}_1 + f(\vec{0}).$$

So the first condition tells us that $\vec{c} \neq \vec{0}$ and the second tells us that $A\vec{e}_1 \neq e_1$. This last equation just means that the first column of A isn't $\vec{e}_1$. We can then choose A and $\vec{c}$ however we want, given these conditions. For instance,

$$f(\vec{x}) := \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \vec{x} + \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$$

will do the trick.

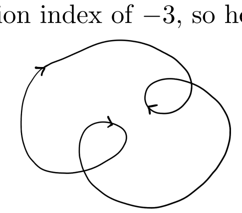
(b) (2 points) Identify two reasons that there does not exist a unit-speed curve $\vec{\alpha}\colon \mathbb{R} \to \mathbb{R}^3$ whose Frenet frame satisfies the equations

$$\begin{pmatrix} \vec{T}' \\ \vec{N}' \\ \vec{B}' \end{pmatrix} = \begin{pmatrix} 0 & s^3 & 0 \\ -s^3 & 0 & \cos s \\ 0 & \cos s & 0 \end{pmatrix} \begin{pmatrix} \vec{T} \\ \vec{N} \\ \vec{B} \end{pmatrix}.$$

Solution: First, the matrix in this equation should be skew-symmetric, but it's not. Also, the matrix here implies that $\kappa(s) = s^3$, but this function is negative for part of the domain of $\vec{\alpha}$, which is impossible.

(c) (2 points) Draw an oriented, closed, planar curve $\vec{\alpha}$ with the property that $\int_{\vec{\alpha}} k(s)\, ds = -6\pi$, where $k(s)$ is the planar curvature. No justification needed.

Solution: Such a curve has a rotation index of -3, so here's an example:



(d) (3 points) Is there a simple, closed curve in the plane with length equal to 6 meters and bounding an area of 3 square meters? Justify your answer.

Solution: No. The isoperimetric inequality tells us that the maximum area enclosed by a simple, closed curve of length 6 meters is

$$\frac{L^2}{4\pi} = \frac{36}{4\pi} = \frac{9}{\pi} \approx 2.86$$

square meters.