Math 4441 **October 5, 2022**

LAST TIME

We defined T_pM as <u>the set of velocity vectors</u> of surface curves through p. This gave T_pM a natural <u>vector space</u> structure. Finally, we gave T_pM an <u>inner product</u> structure.

TODAY

Our primary goal will be to become more familiar with the <u>first fundamental form</u> / <u>metric tensor</u>.

"Differential geometry is just parametrized linear algebra."

—Deane Yang

$$\mathrm{I}_p : T_pM \times T_pM \longrightarrow \mathbb{R}^3$$
$$(\vec{X}, \vec{Y}) \longmapsto \langle \vec{X}, \vec{Y} \rangle_{\mathbb{R}^3}$$

Our first goal is based in linear algebra: how should we package bilinear forms?

Take $\vec{X}, \vec{Y} \in T_pM$. Since $\{\vec{x}_1, \vec{x}_2\}$ is a basis for T_pM,

$$\vec{X} = X^1 \vec{x}_1 + X^2 \vec{x}_2 \quad \& \quad \vec{Y} = Y^1 \vec{x}_1 + Y^2 \vec{x}_2$$

We want a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ so that

$$I_p(\vec{X}, \vec{Y}) = \begin{pmatrix} X^1 \\ X^2 \end{pmatrix}^T \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} Y^1 \\ Y^2 \end{pmatrix},$$

for <u>any</u> choice of $\vec{X}$ & $\vec{Y} \in T_pM$.

$\underline{Q}$ How should we choose $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ to get $I_p(\vec{X}, \vec{Y}) = \begin{pmatrix} X^1 \\ X^2 \end{pmatrix}^T \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} Y^1 \\ Y^2 \end{pmatrix}$, for any $\vec{X}, \vec{Y} \in T_pM$? (5 minutes)

$$I_p(\vec{x}_1, \vec{x}_1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}^T \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}^T \begin{pmatrix} a \\ c \end{pmatrix} = a$$

$$I_p(\vec{x}_1, \vec{x}_2) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}^T \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}^T \begin{pmatrix} b \\ d \end{pmatrix} = b$$

Def. The matrix of metric coefficients for $\vec{x}$ is given by

$$(g_{ij}) = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix},$$

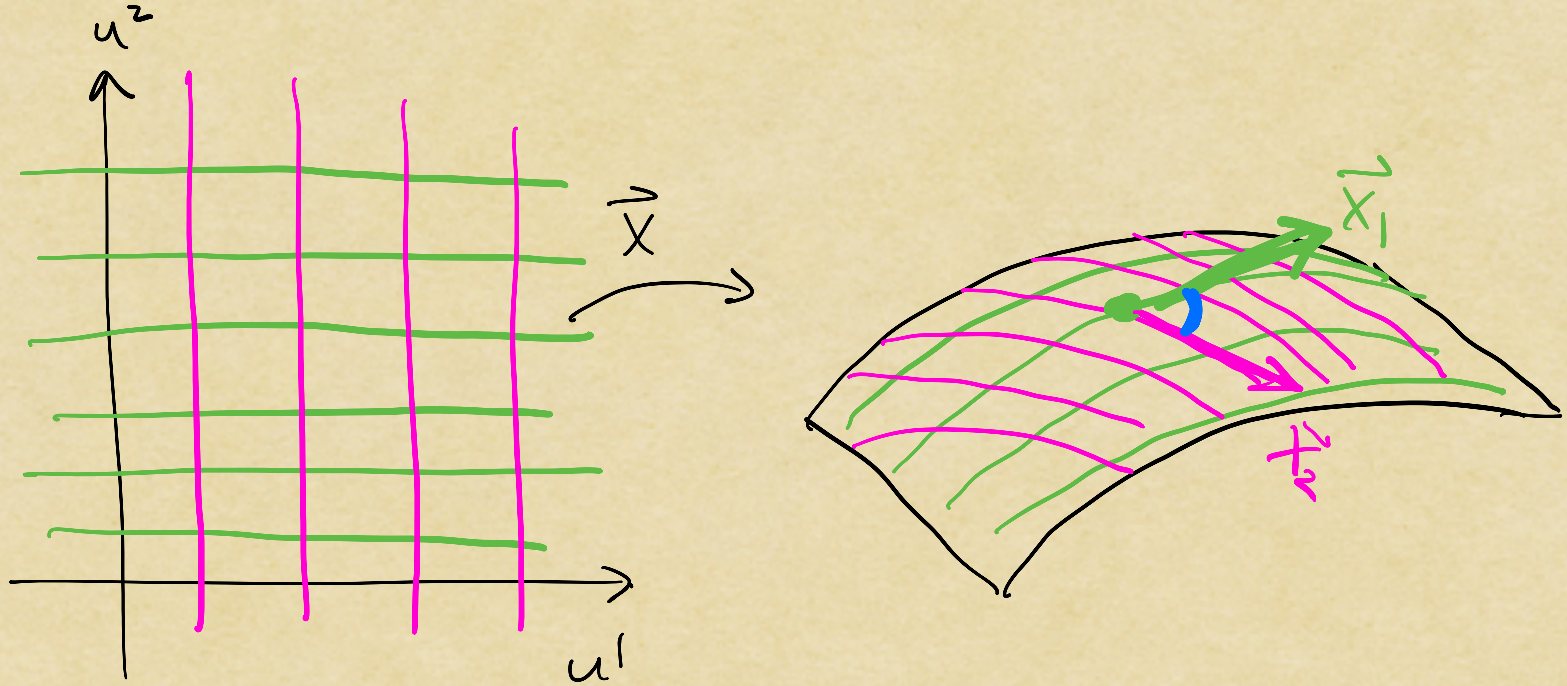
where $g_{ij} : U \longrightarrow \mathbb{R}$ is defined by

$$g_{ij}(u^1, u^2) := \mathrm{I}_{\vec{x}(u^1,u^2)}\left(\vec{x}_i(u^1,u^2), \vec{x}_j(u^1,u^2)\right).$$

Notice that $g_{21} = g_{12}$.

Many sources use the notation

$$E = g_{11}, \quad F = g_{12}, \quad G = g_{22}.$$



u^2
u^1
$\vec{X}$
$\vec{X}_1$
$\vec{X}_2$

We can now derive an important equation:

$$I(\vec{X},\vec{Y}) = \sum_{i,j=1}^{2} X^i Y^j g_{ij}$$

This follows from the matrix version:

$$I(\vec{X},\vec{Y}) = \begin{pmatrix} X^1 \\ X^2 \end{pmatrix}^T \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \begin{pmatrix} Y^1 \\ Y^2 \end{pmatrix}$$

$$= \begin{pmatrix} X^1 \\ X^2 \end{pmatrix}^T \begin{pmatrix} Y^1 g_{11} + Y^2 g_{12} \\ Y^1 g_{21} + Y^2 g_{22} \end{pmatrix}$$

$$= X^1 Y^1 g_{11} + X^1 Y^2 g_{12} + X^2 Y^1 g_{21} + X^2 Y^2 g_{22}$$

Now that we've repackaged the first fundamental form as a matrix, let's use it.

<u>**Def.**</u> Pick $p \in \vec{x}(U)$ and

$$\vec{X} = \sum_i X^i \vec{x}_i, \quad \vec{Y} = \sum_j Y^j \vec{x}_j \quad \in T_p \vec{x}(U).$$

The magnitude of $\vec{X}$ is

$$|\vec{X}| := \sqrt{I_p(\vec{X}, \vec{X})} = \sqrt{\sum_{i,j=1}^{2} X^i X^j g_{ij}}$$

The angle between $\vec{X}$ and $\vec{Y}$ is

$$\measuredangle(\vec{X}, \vec{Y}) := \cos^{-1}\left(\frac{I_p(\vec{X}, \vec{Y})}{|\vec{X}| \cdot |\vec{Y}|}\right) \quad \in [0, \pi].$$

Once we know how to compute lengths of vectors, it's natural to try computing <u>arclengths of surface curves</u>:

"downstairs curve"

Given a surface curve $\vec{\alpha} = \vec{X} \circ \vec{\alpha}_u$, we can compute $L(\vec{\alpha})$ in terms of $\underline{(g_{ij})}$ and $\underline{\vec{\alpha}_u}$:

$$L(\vec{\alpha}) = \int_a^b \|\vec{\alpha}'(t)\| dt = \int_a^b \sqrt{I_{\vec{\alpha}(t)}(\vec{\alpha}'(t), \vec{\alpha}'(t))}\, dt$$

If we write $\vec{\alpha}_u(t) = (\alpha_u^1(t), \alpha_u^2(t))$, then

$$\vec{\alpha}'(t) = d\vec{X}_{\vec{\alpha}_u(t)} \circ \frac{d}{dt}\vec{\alpha}_u(t) = \frac{d}{dt}\alpha_u^1(t) \cdot \vec{X}_1 + \frac{d}{dt}\alpha_u^2(t) \cdot \vec{X}_2$$

Cont'd

$$\vec{\alpha}' = \frac{d\alpha_u^1}{dt}\vec{x}_1 + \frac{d\alpha_u^2}{dt}\vec{x}_2$$

$$L(\vec{\alpha}) = \int_a^b \sqrt{I_{\vec{\alpha}(t)}\left(\vec{\alpha}'(t), \vec{\alpha}'(t)\right)}\, dt$$

$$= \int_a^b \sqrt{\sum_{i,j=1}^{2} \frac{d\alpha_u^i}{dt} \cdot \frac{d\alpha_u^j}{dt}\, g_{ij}}\, dt \quad (\star)$$

Notice that $(\star)$ references <u>two-dimensional</u> data* only. This supports the theme of $\mathrm{im}(\vec{x})$ being a 2D object in $\mathbb{R}^3$.

* $\vec{\alpha}_u$ and (g_{ij})

We'll finish today with some more notation.

The relevance of this notation is intended to be made clearer soon.

Let $\vec{x}$ be a simple surface with metric coefficients (g_{ij}).

$$\boxed{\begin{array}{l}\underline{\text{Def}}.\ \text{The determinant of the matrix of metric} \\ \quad \text{coefficients is} \quad g := \det(g_{ij}).\end{array}}$$

Other notation: $\boxed{(g^{kl}) := (g_{ij})^{-1}}$

Lemma. Let $\vec{X}$ be a simple surface with metric coefficients (g_{ij}).

① $g = \|\vec{X}_1 \times \vec{X}_2\|^2$, so $\sqrt{g}$ gives a scale factor for areas;

② $\begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{pmatrix} = \dfrac{1}{g}\begin{pmatrix} g_{22} & -g_{12} \\ -g_{21} & g_{11} \end{pmatrix}$;

③ $\displaystyle\sum_{k=1}^{2} g_{ik}\, g^{kj} = \delta_i^j = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$, for any i,j.

(Proof.) ① Activity 6. ② Linear alg. formula.
③ Rewrite ②. Do it!

$\diamond$

Next time: How does the first fundamental form* change under coordinate transformation?

*really the matrix of metric coefficients