Math 4441 October 31, 2022

LAST TIME

Given a point p and a unit tangent vector $\vec{u}$, there exists a unique geodesic $\vec{\alpha}$, defined on $(-\varepsilon, \varepsilon)$, for some $\varepsilon > 0$, such that

$$\vec{\alpha}(0) = p \quad \text{and} \quad \vec{\alpha}'(0) = \vec{u}.$$

Why? Picard's theorem

TODAY

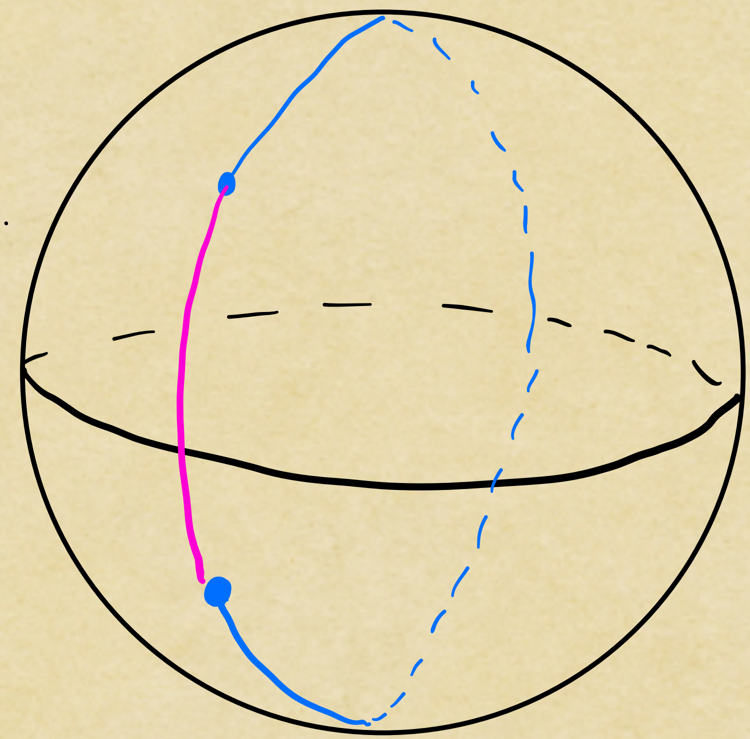
Geodesics do what we want them to do: they locally minimize arclength.

Recall our definition of geodesics:

Def A geodesic is a unit-speed surface curve whose geodesic curvature is everywhere zero.

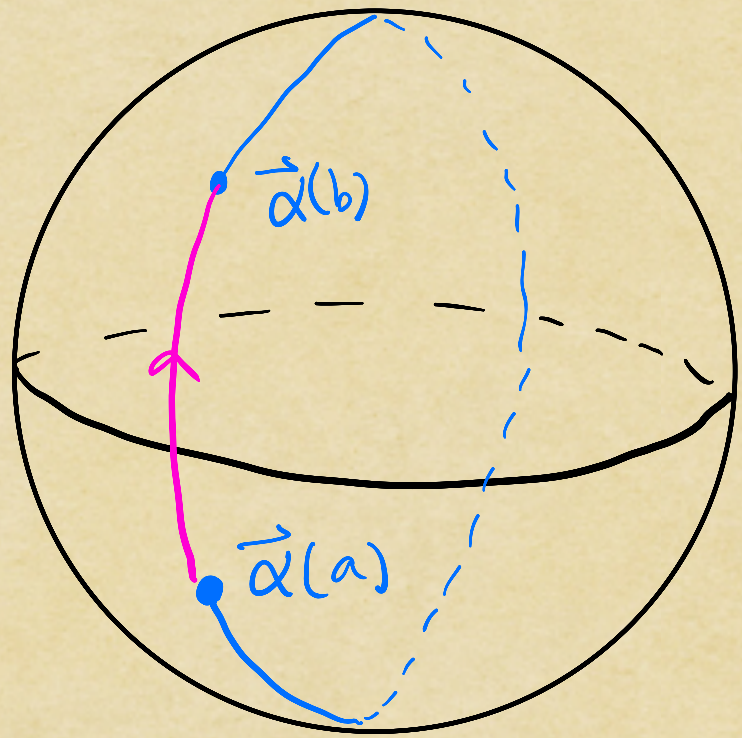
Geodesics are NOT always shortest paths.

But shortest paths are always geodesics!



Thm If $\vec{\alpha} = \vec{X} \circ \vec{\alpha}_u$ is a unit-speed surface curve which minimizes arc length* between points $\vec{\alpha}(a)$ & $\vec{\alpha}(b)$, then $\kappa_g(s) = 0$, for all $s \in (a,b)$.

* i.e., any other surface curve connecting $\vec{\alpha}(a)$ to $\vec{\alpha}(b)$ will have greater arclength btwn these points



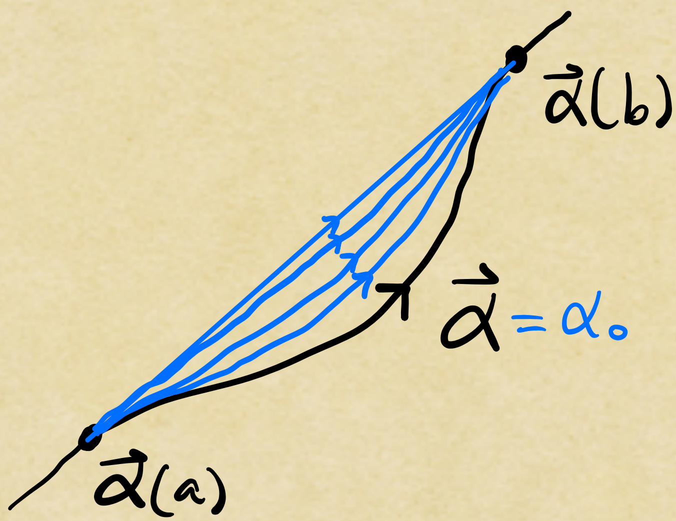
We'll break the proof into two steps:

1. prove the <u>planar</u> version (using <u>calculus of variations</u>);
2. figure out how to recreate the argument on a surface.

<u>Note</u>: Step 2 isn't *quite* as easy as just "<u>pushing everything upstairs</u>."

The planar version

We want to show that if $\vec{\alpha}$ minimizes arclength between $\vec{\alpha}(a)$ & $\vec{\alpha}(b)$, then $0 = \kappa_g(s) = k(s)$ for all $s \in (a,b)$.



How might we get started?

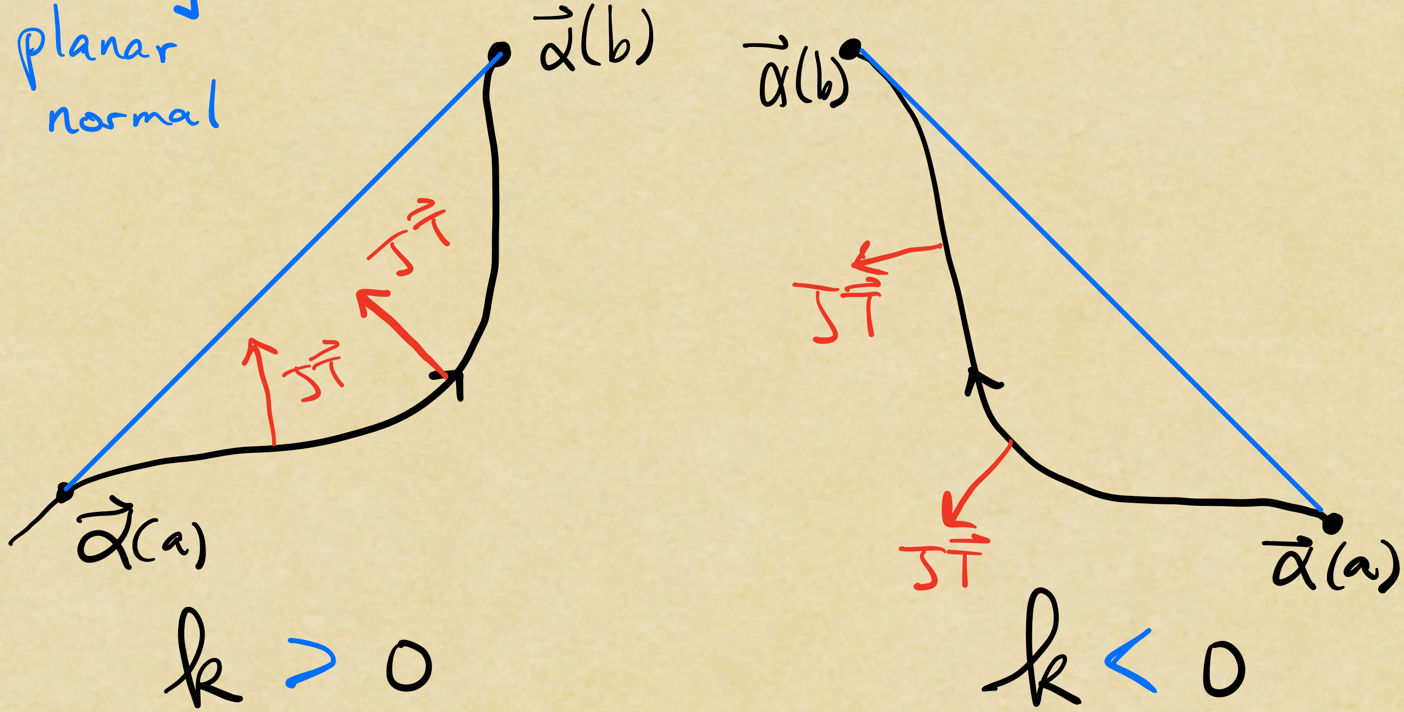
"pull on the ends"

i.e., deform the path to become shorter

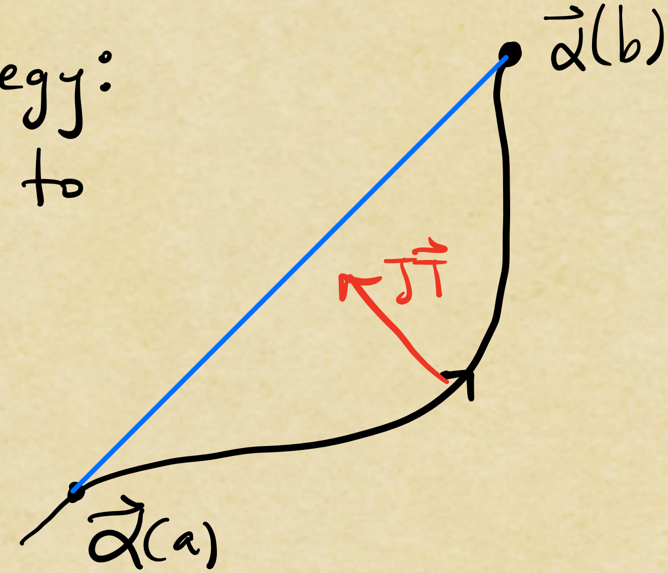
An observation

If $k>0$ or $k<0$, then a shorter path lies on the same side as $\underline{k\vec{n} = kJ\vec{T}}$.

$\underset{\text{planar normal}}{J}$



This leads to our strategy:
Continuously deform $\vec{\alpha}$ to become closer to the straight line.



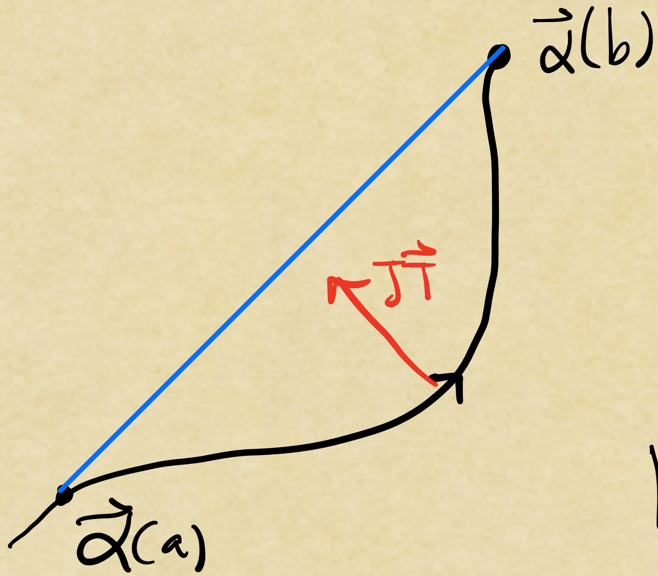
For any $t \in \mathbb{R}$, define a curve

$$\vec{\alpha}_t : (a, b) \longrightarrow \mathbb{R}^2$$

by

$$\vec{\alpha}_t(s) := \vec{\alpha}(s) + t\, k(s) \cdot J\vec{T}(s)$$

Note: $\vec{\alpha}_0 = \vec{\alpha}$.



Since we assumed that $\vec{\alpha}$ is length-minimizing, the function

$$L(t) := \ell(\vec{\alpha}_t) = \int_a^b \|\vec{\alpha}_t'(s)\| \, ds$$

has a minimum at $\underline{t=0}$.

Upshot: $L'(0) = 0$

Calculus of variations

Let's investigate $L'(t)$.

$$L(t) = \int_a^b \|\vec{\alpha}_t'(s)\|\, ds = \int_a^b \sqrt{\left\langle \frac{\partial}{\partial s}\vec{\alpha}_t, \frac{\partial}{\partial s}\vec{\alpha}_t \right\rangle}\, ds$$

$$\therefore\ L'(t) = \frac{\partial}{\partial t}\left(\int_a^b \sqrt{\left\langle \frac{\partial}{\partial s}\vec{\alpha}_t, \frac{\partial}{\partial s}\vec{\alpha}_t \right\rangle}\, ds \right)$$

$$= \int_a^b \frac{\partial}{\partial t}\left(\sqrt{\left\langle \frac{\partial}{\partial s}\vec{\alpha}_t, \frac{\partial}{\partial s}\vec{\alpha}_t \right\rangle} \right) ds$$

$$= \int_a^b \frac{\frac{\partial}{\partial t}\left(\left\langle \frac{\partial}{\partial s}\vec{\alpha}_t, \frac{\partial}{\partial s}\vec{\alpha}_t \right\rangle \right)}{2\sqrt{\left\langle \frac{\partial}{\partial s}\vec{\alpha}_t, \frac{\partial}{\partial s}\vec{\alpha}_t \right\rangle}}\, ds$$

$$L'(t) = \int_a^b \frac{\frac{\partial}{\partial t}\left(\left\langle \frac{\partial}{\partial s}\vec{\alpha}_t, \frac{\partial}{\partial s}\vec{\alpha}_t \right\rangle\right)}{2\sqrt{\left\langle \frac{\partial}{\partial s}\vec{\alpha}_t, \frac{\partial}{\partial s}\vec{\alpha}_t \right\rangle}}\, ds$$

$$= \int_a^b \frac{\cancel{2}\left\langle \frac{\partial^2}{\partial s\,\partial t}\vec{\alpha}_t, \frac{\partial}{\partial s}\vec{\alpha}_t \right\rangle}{\cancel{2}\sqrt{\left\langle \frac{\partial}{\partial s}\vec{\alpha}_t, \frac{\partial}{\partial s}\vec{\alpha}_t \right\rangle}}\, ds$$

Yikes. But we really care about $L'(0)$, where

$$\left.\left\langle \frac{\partial}{\partial s}\vec{\alpha}_t, \frac{\partial}{\partial s}\vec{\alpha}_t \right\rangle\right|_{t=0} = \left\langle \frac{\partial}{\partial s}\vec{\alpha}, \frac{\partial}{\partial s}\vec{\alpha} \right\rangle$$

$$= \langle \vec{\alpha}', \vec{\alpha}' \rangle$$

$$= 1 \quad \left(\begin{array}{l}\text{b/c } \vec{\alpha} = \vec{\alpha}_0 \\ \text{is unit-speed}\end{array}\right)$$

$$\text{So } L'(0) = \int_a^b \left\langle \frac{\partial^2}{\partial s \partial t} \vec{\alpha}_t, \frac{\partial}{\partial s} \vec{\alpha}_t \right\rangle \Big|_{t=0} ds$$

Time for $\underline{\text{integration}}$ $\underline{\text{by parts}}$ = $\underline{\text{undoing the}}$ $\underline{\text{product rule}}$

$$\frac{\partial}{\partial s} \left\langle \frac{\partial}{\partial t} \vec{\alpha}_t, \frac{\partial}{\partial s} \vec{\alpha}_t \right\rangle = \left\langle \frac{\partial^2}{\partial s \partial t} \vec{\alpha}_t, \frac{\partial}{\partial s} \vec{\alpha}_t \right\rangle + \left\langle \frac{\partial}{\partial t} \vec{\alpha}_t, \frac{\partial^2}{\partial s^2} \vec{\alpha}_t \right\rangle$$

$$\therefore \left\langle \frac{\partial^2}{\partial s \partial t} \vec{\alpha}_t, \frac{\partial}{\partial s} \vec{\alpha}_t \right\rangle = \frac{\partial}{\partial s} \left\langle \frac{\partial}{\partial t} \vec{\alpha}_t, \frac{\partial}{\partial s} \vec{\alpha}_t \right\rangle - \left\langle \frac{\partial}{\partial t} \vec{\alpha}_t, \frac{\partial^2}{\partial s^2} \vec{\alpha}_t \right\rangle$$

Finally, recall our definition of $\vec{\alpha}_t$:

$$\vec{\alpha}_t = \vec{\alpha} + t \cdot k \cdot J\vec{T}$$

So $\frac{\partial}{\partial t}\vec{\alpha}_t = k \cdot J\vec{T}$

$$\therefore \left\langle \frac{\partial}{\partial t}\vec{\alpha}_t, \frac{\partial}{\partial s}\vec{\alpha}_t \right\rangle\Big|_{t=0} = \langle k \cdot J\vec{T}, \vec{\alpha}_0' \rangle = 0$$

Then

$$L'(0) = \int_a^b - \left\langle \frac{\partial}{\partial t}\vec{\alpha}_t, \frac{\partial^2}{\partial s^2}\vec{\alpha}_t \right\rangle\Big|_{t=0} ds$$

$$= \int_a^b - \langle k \cdot J\vec{T}, \vec{\alpha}'' \rangle \, ds$$

$$= \int_a^b - k \cdot \overbrace{\langle J\vec{T}, \vec{\alpha}'' \rangle}^{\langle \vec{n}, \frac{d}{ds}\vec{T} \rangle = k} \, ds = \int_a^b - k^2 \, ds$$

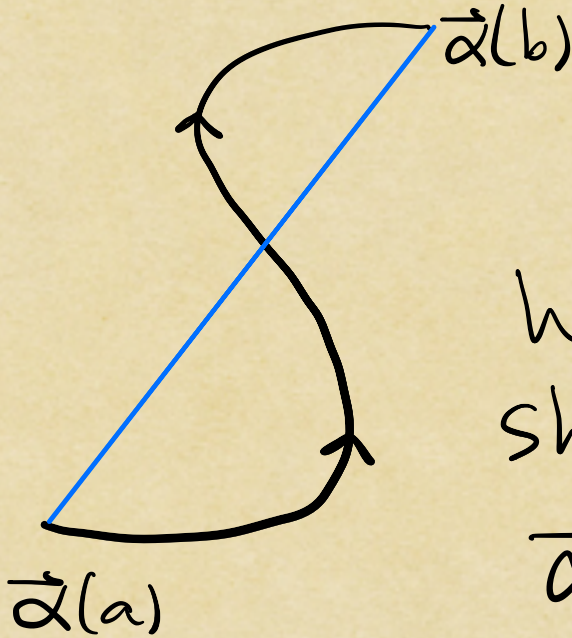
So $L'(0) = \int_a^b -(k(s))^2 \, ds.$

Since $\vec{\alpha}$ is arclength-minimizing, $L'(0) = 0$. At the same time, $(k(s))^2 \geq 0$. So we must have $k(s) = 0$, for all $s \in (a, b)$.

Since $k = \kappa_g$ for planar curves, this means that $\vec{\alpha}$ is a geodesic!

Recap.

The key idea was that if $\kappa_g(s) \neq 0$ for some $s \in (a,b)$, then we could



find a shorter path by pushing towards $\overset{\kappa_g}{k} \cdot J\vec{T}$.

We assumed $\vec{\alpha}$ to be the shortest, built the family

$$\vec{\alpha}_t := \vec{\alpha} + t\,\overset{\kappa_g}{k}\,J\vec{T},$$

and used IBP to find that $\kappa_g \equiv 0$.