Math 4441 — October 3, 2022

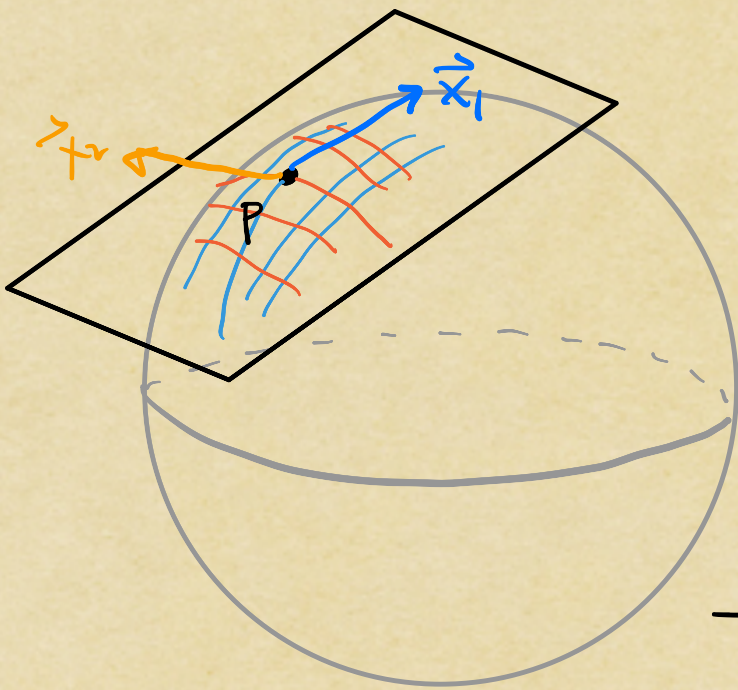
LAST TIME

We defined two geometric invariants of a simple surface: the tangent plane and the* unit normal vector. Specifically, we defined $T_p\vec{x}$ as the plane thru p and is $\perp\ \vec{n}$.

*± 1

TODAY

We'll redefine the tangent plane so that it naturally has the structure of a vector space. In fact, we'll discuss its structure as an inner product space.



Because we want to think of the tangent plane as a tangent space, a natural thing to try is

$$T_p\vec{x} = \text{span}\{\vec{x}_1, \vec{x}_2\}.$$

This doesn't technically work, because the tangent plane may not contain $\vec{0}$

The easiest fix would be to shift the plane to pass thru $\vec{0}$, but we want something more intrinsic

Q How might we give the tangent plane a vector space structure without referring to the origin in $\mathbb{R}^3$?
(1 + 1 minutes)

tangent vector = direction in which we could travel

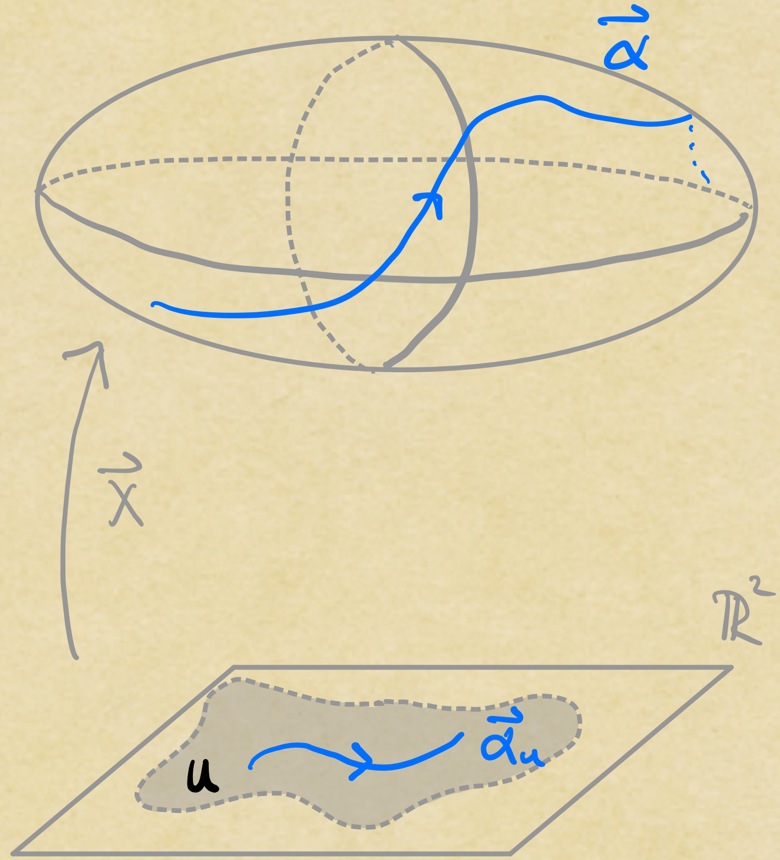
need p to act like an origin.

We'll do something a little strange, but which generalizes well to manifolds.

Def. A **surface curve** $\vec{\alpha}$ on a simple surface $\vec{x}(U)$ is a curve $\vec{\alpha}:(a,b)\to\mathbb{R}^3$ of the form

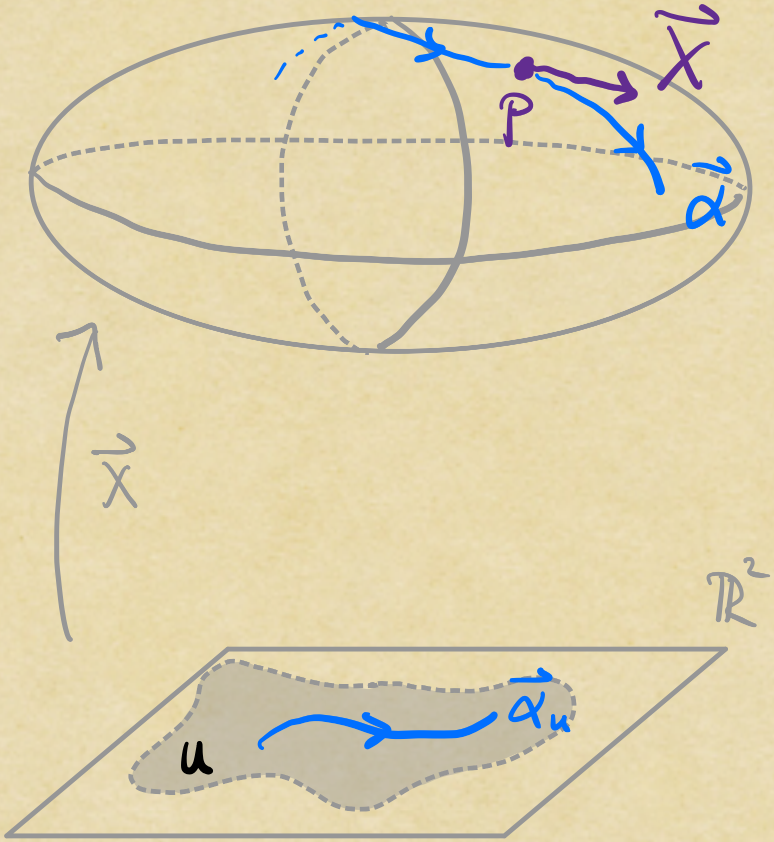
$$\vec{\alpha} = \vec{x}\circ\vec{\alpha}_U,$$

where $\vec{\alpha}_U:(a,b)\to U$ is a curve.



Def. Fix a point p on a simple surface $\vec{x}(U) \subset \mathbb{R}^3$. We call a vector $\vec{X} \in \mathbb{R}^3$ **tangent to $\vec{x}(U)$ at p** if there is a C^1 surface curve $\vec{\alpha}$ with

- $\vec{\alpha}(0) = p$;
- $\vec{\alpha}'(0) = \vec{X}$.



The set of all vectors tangent to $\vec{x}(U)$ at p is denoted $\underline{T_p\vec{x}}$.

Big picture

For an abstract smooth manifold, we know how to take derivatives, so this definition still works.

But in that setting we <u>don't</u> have an ambient $\mathbb{R}^n$ whose vector space structure we can inherit.

How do we know that $T_p\vec{x}$ is a vector space?

Lemma. Let $\vec{x}: U \to \mathbb{R}^3$ be a simple surface. For any $(a,b) \in U$,

$$T_p\vec{x} = \operatorname{im}(d\vec{x}_{(a,b)}),$$

where $p = \vec{x}(a,b) \in \vec{x}(U)$.

Cor For each $p \in \vec{x}(U)$, $T_p U$ is a two-dimensional vector space, with basis given by $\{\vec{x}_1(a,b), \vec{x}_2(a,b)\}$, where $p = \vec{x}(a,b)$.

(Proof of Corollary). We know that $T_p\vec{x} = \mathrm{im}(d\vec{x}_{(a,b)})$ from the lemma.

But <u>the image of a linear map is always a v.s.</u>.

Moreover, the columns of a matrix representing a linear map always <u>span the image</u>

Since $\vec{x}$ is a simple surface, the columns of $d\vec{x}_{(a,b)}$ are <u>linearly independent</u>, so they don't just span — <u>they form a basis.</u>

$\diamond$

(Proof of lemma.) Setup : $\vec{x} : U \longrightarrow \mathbb{R}^3$

$(a,b) \longmapsto p.$

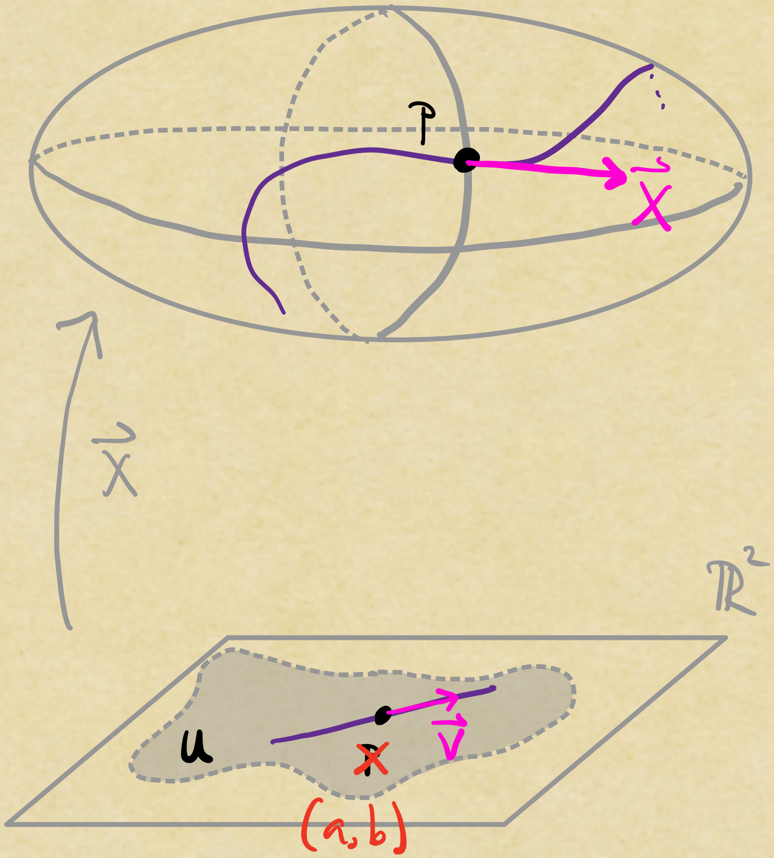
We want to show that $T_p\vec{x} = \operatorname{im}(d\vec{x}_{(a,b)})$.

1 $\operatorname{im}(d\vec{x}_{(a,b)}) \subseteq T_p\vec{x}$

Given $\vec{X} \in \operatorname{im}(d\vec{x}_{(a,b)})$, we need to construct a surface curve $\vec{\alpha}$ with $\vec{\alpha}(0) = p$ & $\vec{\alpha}'(0) = \vec{X}$.

$$\vec{X} \in \operatorname{im}(d\vec{x}_{(a,b)}) \Longrightarrow \vec{X} = d\vec{x}_{(a,b)} \cdot \vec{v},$$

for some $\vec{v} \in \mathbb{R}^2$.



$\vec{\alpha}_U(t) = (a,b) + t\vec{v}$

$\vec{\alpha}_U'(t) = \vec{v}$

Consider the curve

$$\vec{\alpha}_U : (-\varepsilon, \varepsilon) \longrightarrow U$$

$$t \longmapsto (a,b) + t\vec{v},$$

for some small $\varepsilon > 0$. Then $\vec{\alpha} := \vec{x} \circ \vec{\alpha}_U$ satisfies

$$\vec{\alpha}(0) = (\vec{x} \circ \vec{\alpha}_U)(0) = \vec{x}(a,b) = p$$

and

$$\vec{\alpha}'(0) = (\vec{x} \circ \vec{\alpha}_U)'(0) = d\vec{x}_{\vec{\alpha}_U(0)} \cdot \vec{\alpha}_U'(0) = d\vec{x}_{(a,b)} \cdot \vec{v} = \vec{X}.$$

So whenever $\vec{X} \in \mathrm{im}(d\vec{x}_{(a,b)})$, $\vec{X} \in T_p\vec{x}$.

② $T_p\vec{x} \subseteq \mathrm{im}(d\vec{x}_{(a,b)})$

For this direction, we need to check that if $\vec{\alpha}$ is a <u>surface curve with $\vec{\alpha}(0) = p$</u>, then $\vec{\alpha}'(0)$ lies in $\mathrm{im}(d\vec{x}_{(a,b)})$.

In this case, $\vec{\alpha} = \underline{\vec{X} \circ \vec{\alpha}_u}$ for some $\vec{\alpha}_u$ with $\vec{\alpha}_u(0) = (a,b)$.

Check: $\vec{\alpha}'(0) = d\vec{x}_{(a,b)} \cdot \vec{\alpha}_u'(0) \in \mathrm{im}(d\vec{x}_{(a,b)})$ ◇

So where are we now?

1. The "obvious" tangent plane $t_p\vec{x}$ is <u>not a vector space</u>, because it doesn't pass thru $\vec{0}$.

2. The tangent space $T_p\vec{x}$ is <u>a 2D vector space</u>, but only tangent to $\vec{x}(u)$ in an abstract sense.

3. In order to do geometry, we must upgrade $T_p\vec{x}$ to be an <u>inner product space.</u>

Def. The first fundamental form of a surface $M \subset \mathbb{R}^3$ is the collection of inner products defined by

$$\mathrm{I}_p : T_pM \times T_pM \longrightarrow \mathbb{R}$$
$$(\vec{X}, \vec{Y}) \longmapsto \langle \vec{X}, \vec{Y} \rangle_{\mathbb{R}^3}$$

We also call I the metric tensor for M.

Conceptually: $p \rightsquigarrow$ 2D v.s. T_pM

The inner product structure is determined by the way T_pM sits in $\mathbb{R}^3$.

Prop. For any $p \in M$, $I_p : T_pM \times T_pM \to \mathbb{R}$ is a positive-definite, symmetric bilinear form. i.e.,

① $I_p(\vec{X}, \vec{X}) \geq 0$, with $=$ iff $\vec{X} = \vec{0}$;
(positive-definite)

② $I_p(\vec{X}, \vec{Y}) = I_p(\vec{Y}, \vec{X})$ (symmetric);

③ $I_p(a\vec{X} + b\vec{Y}, c\vec{Z} + d\vec{W})$

$$= ac\, I_p(\vec{X}, \vec{Z}) + ad\, I_p(\vec{X}, \vec{W}) + bc\, I_p(\vec{Y}, \vec{Z}) + bd\, I_p(\vec{Y}, \vec{W})$$

(bilinearity)

(Proof). We inherited I_p from $\langle -, - \rangle$ on $\mathbb{R}^3$. $\diamond$

"Differential geometry is just parametrized linear algebra."

—Deane Yang