Math 4441 October 26, 2022

LAST TIME

Geodesic curvature is intrinsic, but normal curvature is not.

TODAY

Given any point on a surface and any chosen direction, we can always travel in that direction in a manner that is as "straight" as possible.

Recall (Activity 7) $\kappa^2 = \kappa_n^2 + \kappa_g^2$

$= (\kappa \sin\theta)^2 + (\kappa \cos\theta)^2$

So a surface curve has two curvatures:

κ_n : due to the surface

κ_g : felt by the surface

This motivates:

Def A geodesic is a unit-speed surface curve whose geodesic curvature is everywhere zero.

Let's write down two ODEs for a geodesic.

Prop. Let $\vec{\alpha} = \vec{x} \circ \vec{\alpha}_u$ be a unit-speed surface curve. Then the following are equivalent:

1. $\vec{\alpha}$ is a geodesic;
2. $\langle \vec{n}, \vec{T} \times \vec{T}' \rangle \equiv 0$;
3. for $k = 1, 2$, $(\alpha_u^k)'' + \sum_{i,j=1}^{2} (\alpha_u^i)'(\alpha_u^j)' \Gamma_{ij}^k \equiv 0$.

(Proof.) 1 $\Leftrightarrow$ 3 follows from our intrinsic formula.

For 1 $\Leftrightarrow$ 2, we have

$$\kappa_g = \langle \vec{T}', \vec{S} \rangle = \underbrace{\langle \vec{T}', \vec{n} \times \vec{T} \rangle = \langle \vec{n}, \vec{T} \times \vec{T}' \rangle}_{\text{scalar triple product}}. \quad \diamond$$

The ODEs can be helpful for computations, but here's a characterization you can see a bit more easily.

Prop. A unit-speed surface curve $\vec{\alpha} = \vec{x} \circ \vec{\alpha}_u$ is a geodesic iff $\vec{\alpha}''$ is everywhere $\perp$ to $\vec{x}$.

(Proof.) We showed that $\vec{\alpha}$ unit-speed gives

$$\vec{\alpha}'' = \kappa_n \vec{n} + \kappa_g \vec{S},$$

So $\vec{\alpha}$ is a geodesic iff $\vec{\alpha}'' = \kappa_n \vec{n} \perp \vec{x}$. $\diamond$

Upshot: As far as an inhabitant of $\vec{x}$ can tell, a geodesic has no acceleration — its tangent vector never changes.

$\vec{T}$ is a parallel vector field!

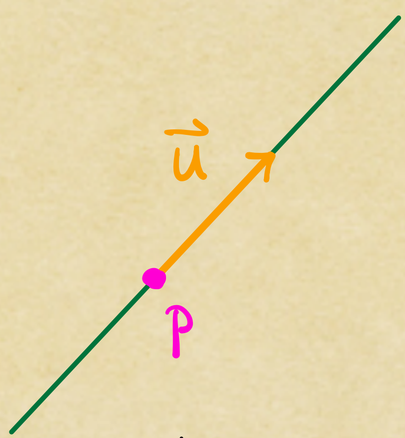
Example In activity 7, you showed that planar curves satisfy $\kappa_n = \underline{0}$. What does this mean about geodesics on the plane?

geodesic $\Rightarrow \kappa_g = 0$

$\kappa^2 = \kappa_n^2 + \kappa_g^2 = 0^2 + 0^2 = 0 \Rightarrow$ space curvature

must be a straight line.

Q Given a point $p \in \mathbb{R}^2$ and a unit vector $\vec{u}$ at p, we can always draw a unique straight line thru p which is parallel to $\vec{u}$.

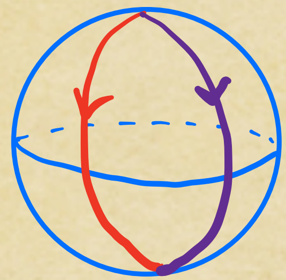


Can we replace "$\mathbb{R}^2$" with "$\vec{X}$" and "straight line" with "geodesic"?

What problems might we encounter?

Thoughts on existence & uniqueness of geodesics

given two points, maybe no unique geodesic btwn them



existence fails, too.

but given a point + a vector, we have an initial value problem.

geodesic ODEs + initial value p + initial derivative $\vec{u}$

Thm (Existence & uniqueness of geodesics)

Let $\vec{x}: U \to \mathbb{R}^3$ be a simple surface, let $p \in \operatorname{im} \vec{x}$ be a point on $\vec{x}$, and let $\vec{V} \in T_p\vec{x}$ be a unit-length vector. Then

① there is some $\varepsilon > 0$ and geodesic

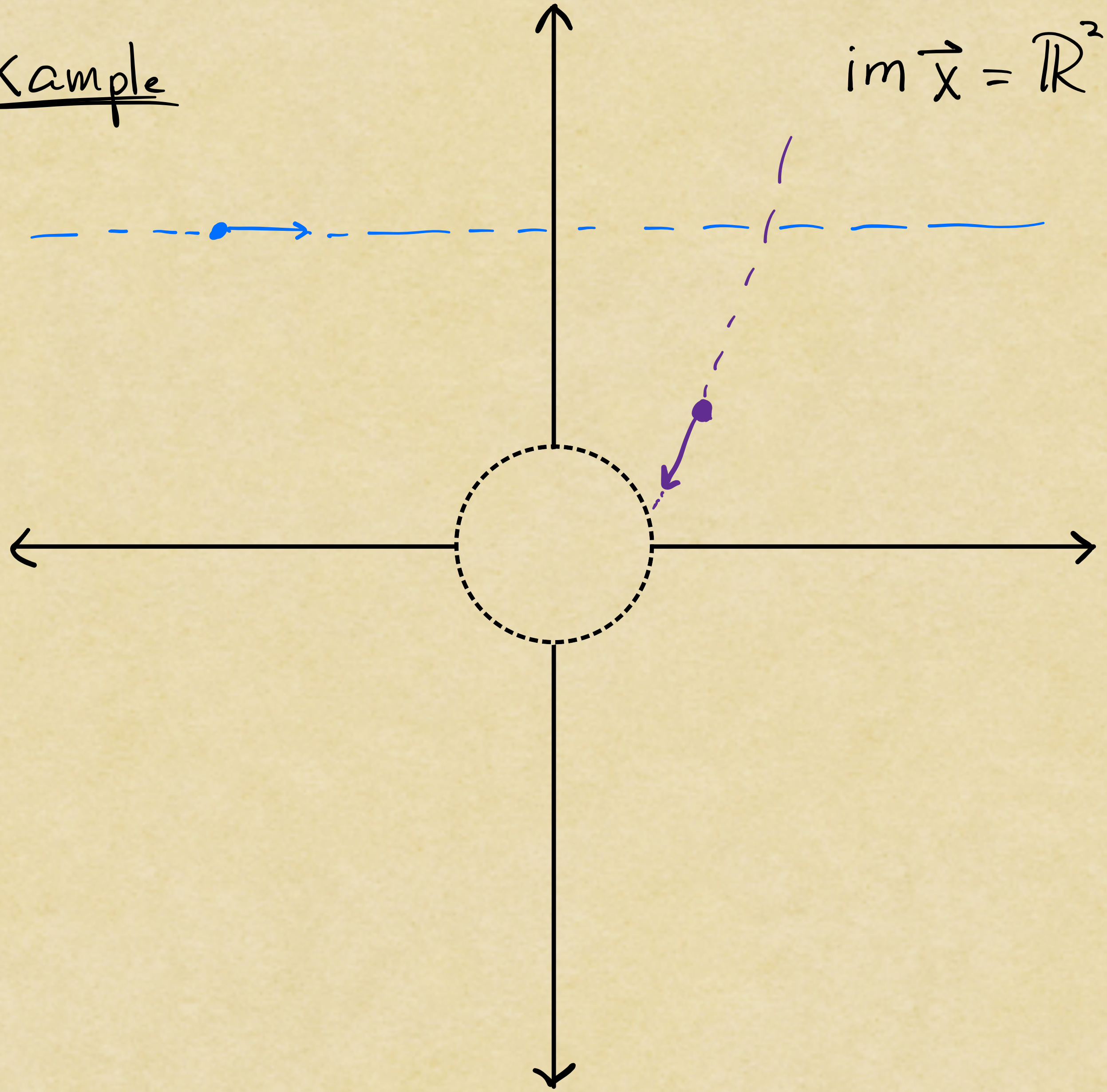
$$\vec{\alpha}: (-\varepsilon, \varepsilon) \longrightarrow \vec{x}(U)$$

with $\underline{\vec{\alpha}(0) = p}$ and $\underline{\vec{\alpha}'(0) = \vec{V}}$;

② if $\vec{\beta}: (-\delta, \delta) \to \vec{x}(U)$ is some other geodesic with $\underline{\vec{\beta}(0) = p}$ and $\underline{\vec{\beta}'(0) = \vec{V}}$, then $\underline{\vec{\beta}(s) = \vec{\alpha}(s)}$, for $|s| < \varepsilon, \delta$.

Example

$\text{im}\,\vec{x} = \mathbb{R}^2 - \text{disc}$



(Proof sketch.) We can express our situation as an initial value problem downstairs.

$$\vec{\alpha} : (-\varepsilon, \varepsilon) \to \vec{X} \quad \rightsquigarrow \quad \vec{\alpha} = \vec{X} \circ \vec{\alpha}_u$$

$$\vec{\alpha}(0) = p \quad \rightsquigarrow \quad \vec{\alpha}_u(0) = \vec{X}^{-1}(p) \in U.$$

$$\vec{\alpha}'(0) = \vec{V} \quad \rightsquigarrow \quad \vec{\alpha}_u'(0) = (d\vec{x}^{-1})_p(\vec{V})$$

$$\left(\text{OR write } \vec{V} = V^1 \vec{x}_1 + V^2 \vec{x}_2 \rightsquigarrow \vec{\alpha}_u'(0) = \begin{pmatrix} V^1 \\ V^2 \end{pmatrix}\right)$$

$$k_g \equiv 0 \quad \rightsquigarrow \quad (\alpha_u^k)'' + \sum_{i,j=1}^{2} (\alpha_u^i)'(\alpha_u^j)' \Gamma_{ij}^k = 0$$

for $k = 1, 2$.

By Picard's theorem on existence and uniqueness of solutions to ODEs, some such

$$\vec{\alpha}_u : (-\varepsilon, \varepsilon) \to \mathbb{R}^2$$

exists.

But! We also need $\vec{\alpha}$ to be <u>unit-speed</u>.

Perhaps surprisingly, this is forced by the ODEs, though the computation is too long to include here. $\diamond$