Math 4441 October 24, 2022

LAST TIME

With great pain, we found formulas for κ_n and κ_g in terms of $\vec{\alpha}_u$, $\vec{x}$, and their derivatives.

Point: Separate surface contributions from downstairs curve contributions.

THESE SLIDES

We can write κ_g in terms of (g_{ij}) and $\vec{\alpha}_u$, so geodesic curvature is intrinsic — it can be computed without reference to $\mathbb{R}^3$.

The formulas we have

$$\kappa_g \vec{S} = \sum_{k=1}^{2}\left[(\alpha_u^k)'' + \sum_{i,j=1}^{2}(\alpha_u^i)'(\alpha_u^j)'\Gamma_{ij}^k\right]\vec{x}_k \quad \& \quad \kappa_n = \sum_{i,j=1}^{2}(\alpha_u^i)'(\alpha_u^j)' L_{ij}$$

So κ_g depends on $\vec{\alpha}_u$ and $\underline{\Gamma_{ij}^k}$, while

κ_n depends on $\vec{\alpha}_u$ and $\underline{L_{ij}}$.

So κ_g will be intrinsic if <u>the Christoffel symbols are intrinsic</u>.

Recall: $\Gamma_{ij}^k := \sum_{\ell=1}^{2} \langle \vec{x}_{ij}, \vec{x}_\ell \rangle g^{k\ell}$

only depends on $\vec{x}$.

Proposition. The Christoffel symbols satisfy

$$\Gamma_{ij}^{k} = \frac{1}{2} \sum_{\ell=1}^{2} g^{\ell k} \left[\frac{\partial g_{i\ell}}{\partial u^j} - \frac{\partial g_{ij}}{\partial u^\ell} + \frac{\partial g_{j\ell}}{\partial u^i} \right],$$

and are thus <u>intrinsic</u>.

(Proof.) Not hard, but need to skip for time. ◇

<u>Note</u>: The formula itself isn't that important — the point is that <u>it exists at all!</u>

Corollary The geodesic curvature is given by

$$\kappa_g = \sqrt{g}\Big[(\alpha_u^1)'(\alpha_u^2)'' - (\alpha_u^1)''(\alpha_u^2)' + \sum_{i,j=1}^{2}\Big((\alpha_u^1)'\Gamma_{ij}^2 - \Gamma_{ij}^1(\alpha_u^2)'\Big)(\alpha_u^i)'(\alpha_u^j)'\Big],$$

and is thus <u>intrinsic</u>.

(Proof.) Another skipped computation. ◇

Remarks

1. Again, the point is that <u>a formula exists</u>.
2. Normal curvature is <u>not intrinsic</u> b/c <u>L_{ij}</u> depends on $\vec{x}$.