Math 4441 October 19, 2022

LAST TIME

We defined the normal and geodesic curvatures, and began looking for formulas for them in terms of $\vec{x}$, $\vec{\alpha}_u$, and their derivatives

Idea: Split $\vec{\alpha}''$ into its "normal" and "geodesic" parts.

TODAY Keep Computing.

Big picture / long term: only want things to depend on (g_{ij}) and $\vec{\alpha}_u$, if possible.

Four steps:

1. Express $\vec{\alpha}''$ in terms of $\vec{\alpha}_u$ and $\vec{x}$ and their derivatives.

   Done: $\vec{\alpha}'' = \sum_{i=1}^{2} (\alpha_u^i)'' \vec{x}_i + \sum_{i,j=1}^{2} (\alpha_u^i)' (\alpha_u^j)' \vec{x}_{ij}$

2. Wild new notation / packaging for the second derivatives of $\vec{x}$.

3. Use this new notation to express $\vec{x}_{ij}$ in the basis $\{\vec{n}, \vec{x}_1, \vec{x}_2\}$.

4. Combine everything into formulas for κ_n & κ_g.

② Wild new notation / packaging for the <u>second derivatives of $\vec{x}$</u>.

Based on ①, decomposing $\vec{\alpha}''$ into normal and tangential ($\approx$ <u>geodesic</u>) components will require decomposing <u>$\vec{x}_{ij}$</u>

So we want to write

$$\vec{x}_{ij} = \underbrace{?? \ \underline{\vec{x}_1} + ?? \ \underline{\vec{x}_2}}_{\text{tangential}} + \underbrace{?? \ \underline{\vec{n}}}_{\text{normal}}$$

Note: We do not necessarily have an <u>ONB</u>.

② Wild new notation / packaging for the second derivatives of $\vec{x}$.

Our new notation will give away the story.

Def'n The coefficients of the second fundamental form for $\vec{x}$ are the functions $L_{ij} : U \to \mathbb{R}$ def'd by

$$L_{ij} := \langle \vec{x}_{ij}, \vec{n} \rangle ,$$

for $1 \leq i, j \leq 2$.

For now this is just notation.

Later: A bilinear form $T_pM \times T_pM \to \mathbb{R}$ rep'd by L_{ij}.

② Wild new notation / packaging for the second derivatives of $\vec{x}$.

Def The Christoffel symbols of $\vec{x}: U \to \mathbb{R}^3$ are the eight functions $\Gamma_{ij}^k : U \longrightarrow \mathbb{R}$ def'd by

$$\Gamma_{ij}^k := \sum_{\ell=1}^{2} \langle \vec{x}_{ij}, \vec{x}_\ell \rangle \, g^{\ell k},$$

for $1 \le i, j, k \le 2$.

This definition is gross! But remember where (we're hoping) it comes from: Γ_{ij}^k should be the coefficient on $\vec{x}_k$ in an expansion of $\vec{x}_{ij}$.

Four steps:

1. Express $\vec{\alpha}''$ in terms of $\vec{\alpha}_u$ and $\vec{x}$ and their derivatives. $\vec{\alpha}'' = \sum_{i=1}^{2} (\alpha_u^i)'' \vec{x}_i + \sum_{i,j=1}^{2} (\alpha_u^i)' (\alpha_u^j)' \vec{x}_{ij}$

2. Wild new notation / packaging for the second derivatives of $\vec{x}$.

   Done: $L_{ij} := \langle \vec{x}_{ij}, \vec{n} \rangle$ & $\Gamma_{ij}^{k} = \sum_{\ell=1}^{2} \langle \vec{x}_{ij}, \vec{x}_\ell \rangle g^{k\ell}$

3. Use this new notation to express $\vec{x}_{ij}$ in the basis $\{\vec{n}, \vec{x}_1, \vec{x}_2\}$.

4. Combine everything into formulas for κ_n & κ_g.

③ Use this new notation to express $\vec{x}_{ij}$ in the basis $\{\vec{n}, \vec{x}_1, \vec{x}_2\}$.

A note on bases: given a surface curve $\vec{\alpha} = \vec{x} \circ \vec{\alpha}_u$, we have three useful bases.

basis	name	depends on
$\{\vec{T}, \vec{N}, \vec{B}\}$	Frenet frame	just the curve
$\{\vec{T}, \vec{S}, \vec{n}\}$	Darboux frame	curve + surface
$\{\vec{n}, \vec{x}_1, \vec{x}_2\}$ *	surface basis	just the surface

* NOT orthonormal

③ Use this new notation to express $\vec{x}_{ij}$ in the basis $\{\vec{n}, \vec{x}_1, \vec{x}_2\}$.

Proposition Let $\vec{x}: U \to \mathbb{R}^3$ be a simple surface. Then

$$\vec{x}_{ij} = L_{ij}\vec{n} + \sum_{k=1}^{2} \Gamma_{ij}^{k} \vec{x}_k,$$

for any $1 \leq i, j \leq 2$.

This is why we care about Γ_{ij}^{k}.

(Proof) For a chosen pair $1 \leq i, j \leq 2$, we know we can write

$$\vec{x}_{ij} = A\vec{n} + B_1\vec{x}_1 + B_2\vec{x}_2$$

for some A, B_1, B_2.

③ Use this new notation to express $\vec{x}_{ij}$ in the basis $\{\vec{n}, \vec{x}_1, \vec{x}_2\}$.

From $\vec{x}_{ij} = A\,\vec{n} + B_1\vec{x}_1 + B_2\vec{x}_2$ we get A:

$$\langle \vec{x}_{ij}, \vec{n} \rangle = \langle A\,\vec{n} + B_1\vec{x}_1 + B_2\vec{x}_2, \vec{n} \rangle$$

$$= A.$$

But this is just $\underline{L_{ij}}$!

So we have the right coefficient for $\vec{n}$.

For B_k ($k = 1$ or 2),

$$\langle \vec{x}_{ij}, \vec{x}_k \rangle = \langle A\,\vec{n} + B_1\vec{x}_1 + B_2\vec{x}_2, \vec{x}_k \rangle$$

$$= B_1 \langle \vec{x}_1, \vec{x}_k \rangle + B_2 \langle \vec{x}_2, \vec{x}_k \rangle$$

$$= \sum_{\ell=1}^{2} B_\ell\, g_{\ell k}$$

③ Use this new notation to express $\vec{x}_{ij}$ in the basis $\{\vec{n}, \vec{x}_1, \vec{x}_2\}$.

So $\langle \vec{x}_{ij}, \vec{x}_k \rangle = B_1 g_{1k} + B_2 g_{2k}$, for $k = 1, 2$.

Thus,
$$\begin{pmatrix} \langle \vec{x}_{ij}, \vec{x}_1 \rangle \\ \langle \vec{x}_{ij}, \vec{x}_2 \rangle \end{pmatrix} = \begin{pmatrix} B_1 g_{11} + B_2 g_{21} \\ B_1 g_{12} + B_2 g_{22} \end{pmatrix} = \begin{pmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}.$$

Now we solve this matrix equation for $\begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$:
$$\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = \begin{pmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{pmatrix}^{-1} \begin{pmatrix} \langle \vec{x}_{ij}, \vec{x}_1 \rangle \\ \langle \vec{x}_{ij}, \vec{x}_2 \rangle \end{pmatrix} = \begin{pmatrix} g^{11} & g^{12} \\ g^{12} & g^{22} \end{pmatrix} \begin{pmatrix} \langle \vec{x}_{ij}, \vec{x}_1 \rangle \\ \langle \vec{x}_{ij}, \vec{x}_2 \rangle \end{pmatrix}$$

So
$$B_k = \langle \vec{x}_{ij}, \vec{x}_1 \rangle g^{k1} + \langle \vec{x}_{ij}, \vec{x}_2 \rangle g^{k2}$$
$$= \sum_{\ell=1}^{2} \langle \vec{x}_{ij}, \vec{x}_\ell \rangle g^{k\ell} = \Gamma_{ij}^{k}.$$

$\diamond$

Four steps:

1. Express $\vec{\alpha}''$ in terms of $\vec{\alpha}_u$ and $\vec{x}$ and their derivatives. $\vec{\alpha}'' = \sum_{i=1}^{2} (\alpha_u^i)'' \vec{x}_i + \sum_{i,j=1}^{2} (\alpha_u^i)'(\alpha_u^j)' \vec{x}_{ij}$

2. Wild new notation / packaging for the second derivatives of $\vec{x}$.

   $L_{ij} := \langle \vec{x}_{ij}, \vec{n} \rangle$

   $\Gamma_{ij}^{k} = \sum_{\ell=1}^{2} \langle \vec{x}_{ij}, \vec{x}_\ell \rangle g^{k\ell}$

3. Use this new notation to express $\vec{x}_{ij}$ in the basis $\{\vec{n}, \vec{x}_1, \vec{x}_2\}$.

   Done: $\vec{x}_{ij} = L_{ij}\vec{n} + \sum_{k=1}^{2} \Gamma_{ij}^{k} \vec{x}_k$

4. Combine everything into formulas for κ_n & κ_g.

④ Combine everything into formulas for κ_n & κ_g.

We now have two ways of writing $\vec{\alpha}''$:

Ⓐ $\vec{\alpha}'' = \underline{\kappa_n}\,\vec{n} + \underline{\kappa_g}\,\vec{S}$ ← tangent

Ⓑ $\vec{\alpha}'' = \sum_{i=1}^{2} (\alpha_u^i)'' \vec{x}_i + \sum_{i,j=1}^{2} (\alpha_u^i)'(\alpha_u^j)' \vec{x}_{ij}$ ← neither tangent nor perpendicular

We can decompose $\vec{x}_{ij}$ into normal & tangential parts:

$$\vec{x}_{ij} = L_{ij}\vec{n} + \sum_{k=1}^{2} \Gamma_{ij}^{k} \vec{x}_k.$$

Subbing into Ⓑ:

$$\vec{\alpha}'' = \sum_{i=1}^{2} (\alpha_u^i)'' \vec{x}_i + \sum_{i,j=1}^{2} (\alpha_u^i)'(\alpha_u^j)' L_{ij}\vec{n} + \sum_{i,j,k=1}^{2} (\alpha_u^i)'(\alpha_u^j)' \Gamma_{ij}^{k} \vec{x}_k$$

$$= \left[\sum_{i,j=1}^{2} (\alpha_u^i)'(\alpha_u^j)' L_{ij}\right]\vec{n} + \sum_{k=1}^{2}\left[(\alpha_u^k)'' + \sum_{i,j=1}^{2} (\alpha_u^i)'(\alpha_u^j)' \Gamma_{ij}^{k}\right]\vec{x}_k$$

(4) Combine everything into formulas for κ_n & κ_g.

Altogether,

$$\vec{\alpha}'' = \kappa_n \vec{n} + \kappa_g \vec{S}$$

&

$$\vec{\alpha}'' = \left[\sum_{i,j=1}^{2} (\alpha_u^i)'(\alpha_u^j)' L_{ij}\right]\vec{n} + \sum_{k=1}^{2}\left[(\alpha_u^k)'' + \sum_{i,j=1}^{2}(\alpha_u^i)'(\alpha_u^j)'\Gamma_{ij}^k\right]\vec{x}_k.$$

Equating normal and tangential components:

$$\kappa_n = \sum_{i,j=1}^{2}(\alpha_u^i)'(\alpha_u^j)' L_{ij}$$

&

$$\kappa_g \vec{S} = \sum_{k=1}^{2}\left[(\alpha_u^k)'' + \sum_{i,j=1}^{2}(\alpha_u^i)'(\alpha_u^j)'\Gamma_{ij}^k\right]\vec{x}_k$$

Yikes. These are scary, but they accomplish the goal: express κ_n and κ_g in terms of $\vec{x}$, $\vec{\alpha}_u$, and their derivatives.

④ Combine everything into formulas for κ_n & κ_g.

$$\kappa_n = \sum_{i,j=1}^{2} (\alpha_u^i)'(\alpha_u^j)' L_{ij}$$

$$\kappa_g \vec{S} = \sum_{k=1}^{2} \left[(\alpha_u^k)'' + \sum_{i,j=1}^{2} (\alpha_u^i)'(\alpha_u^j)' \Gamma_{ij}^k \right] \vec{x}_k .$$

These equations can be hard to appreciate at first, but remember that we want to know what an inhabitant of $\vec{x}$ can "see."

These equations tell us that

seeing κ_n = seeing L_{ij}

seeing κ_g = seeing Γ_{ij}^k

So we need to study L_{ij} & Γ_{ij}^k some more.