Math 4441 October 12, 2022

LAST TIME

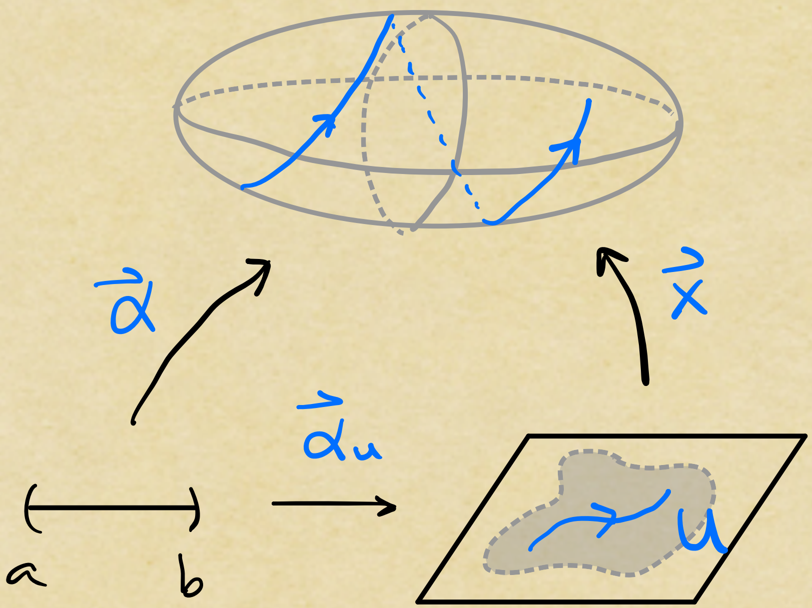
How does the matrix of metric coefficients behave under coordinate transformations?

TODAY

We begin studying surface curves from the perspective of $\vec{x}$.

Idea: Locally, surface curves seem kind of planar.

Throughout today, we'll have a surface curve:



Our goal is to characterize the geometry of $\vec{\alpha}$ w.r.t. $\vec{x}$.

Discussing curvature will require second derivatives: $\vec{X}_{ij} := \frac{\partial^2 \vec{x}}{\partial u^j \partial u^i}$.

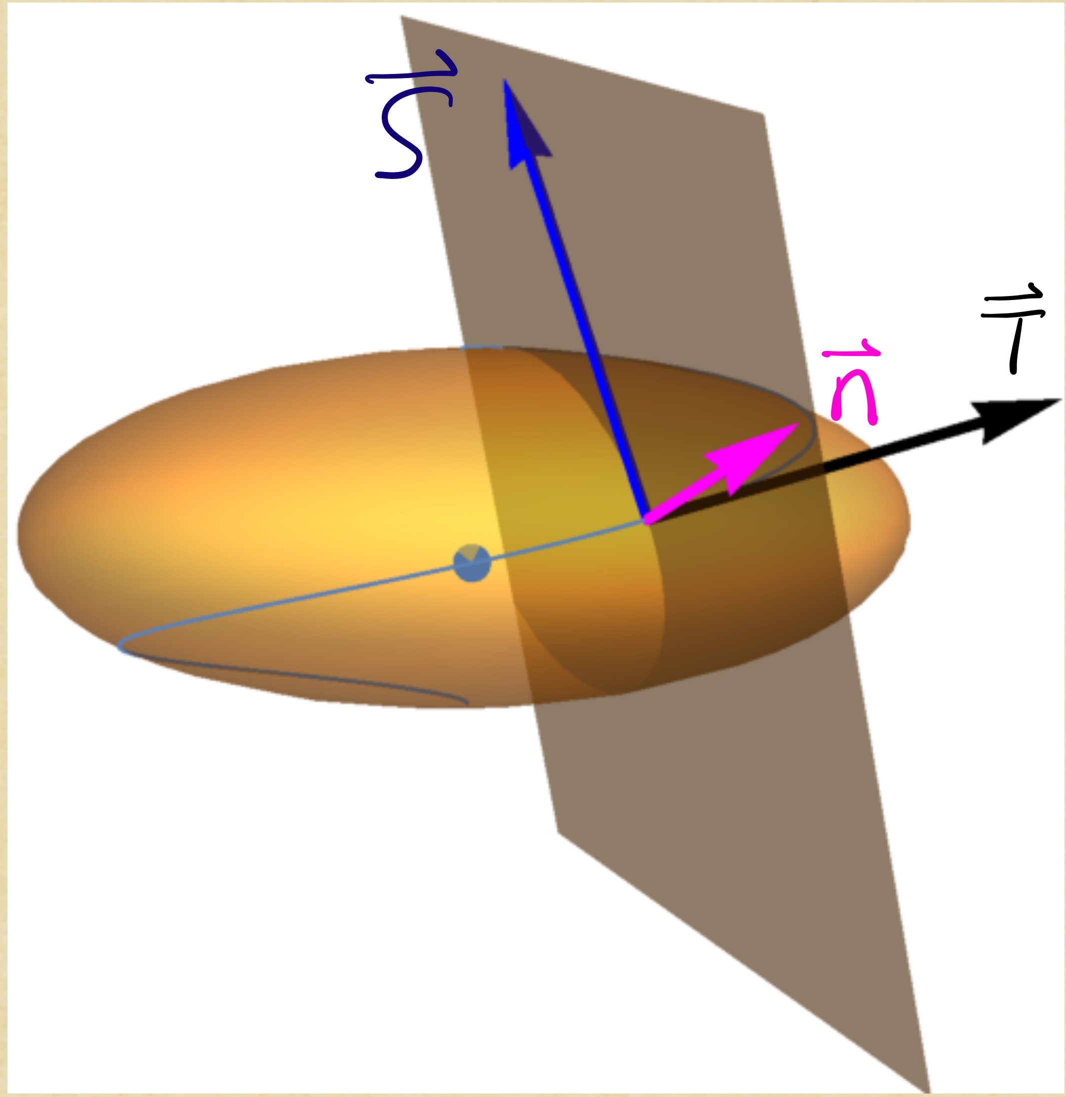
Clairaut's theorem: $\vec{X}_{ji} = \vec{X}_{ij}$

How did we measure curvature before?

<u>Space curves</u>: What is the <u>magnitude</u> of the change in <u>$\vec{T}(s)$</u> as we move along $\vec{\alpha}$?

<u>Planar curves</u>: We have a <u>preferred normal vector</u>, so we can ask how much $\vec{T}(s)$ changes <u>in the direction of this normal</u>?

With surface curves, we have two natural normal vectors:



- (surface normal)

$$\vec{n} := \frac{\vec{x}_1 \times \vec{x}_2}{\|\vec{x}_1 \times \vec{x}_2\|}$$

- (intrinsic normal)

$$\vec{S} := \vec{n} \times \vec{T}$$

from $\vec{x}$ from $\vec{\alpha}$

With two different normal vectors, we can compute two different curvatures:

Def. Given a unit-speed surface curve

$$\vec{\alpha}(s) = (\vec{x} \circ \vec{\alpha}_u)(s),$$

define the

- <u>normal curvature</u> $\kappa_n(s) := \left\langle \frac{d}{ds}\vec{T}(s), \vec{n}(\vec{\alpha}_u(s)) \right\rangle$
- <u>geodesic curvature</u> $\kappa_g(s) := \left\langle \frac{d}{ds}\vec{T}(s), \vec{S}(s) \right\rangle$

The normal and geodesic curvatures tell us the <u>coefficients</u> of $\frac{d}{ds}\vec{T}(s)$ in the basis

$$\{\vec{T}(s), \vec{S}(s), \vec{n}(s)\}.$$

This is an ONB, so

$$\frac{d}{ds}\vec{T} = \underbrace{\left\langle \frac{d}{ds}\vec{T}, \vec{T} \right\rangle}_{\substack{=0,\ b/c \\ \|\vec{T}\| \equiv 1}} \vec{T} + \underbrace{\left\langle \frac{d}{ds}\vec{T}, \vec{S} \right\rangle}_{k_g} \vec{S} + \underbrace{\left\langle \frac{d}{ds}\vec{T}, \vec{n} \right\rangle}_{k_n} \vec{n}$$

You'll explore this basis again on Friday.

What do κ_n and κ_g measure?

κ_n : the extent to which $\vec{\alpha}$ is forced to curve b/c it lives in $\vec{x}$

κ_g : the extent to which $\vec{\alpha}$ appears to curve from the perspective of $\vec{x}$

<u>New goal</u>: Find formulas for κ_n and κ_g which use $\vec{x}$ and $\vec{\alpha}_u$ (i.e., formulas expressed in index form)

Long-term, we'd really like formulas which only use 2D data like (g_{ij}) and $\vec{\alpha}_u$.

Idea: This data is intrinsic, in the sense that it's visible to an inhabitant of the surface.

Four steps: Curvature is "order 2"

1. Express $\vec{\alpha}''$ in terms of $\vec{\alpha}_u$ and $\vec{x}$ and their derivatives.

2. Wild new notation / packaging for the second derivatives of $\vec{x}$.

3. Use this new notation to express $\vec{x}_{ij}$ in the basis $\{\vec{n}, \vec{x}_1, \vec{x}_2\}$.

4. Combine everything into formulas for κ_n & κ_g.

~~Today~~ In these slides we'll just do 1.

① Express $\vec{\alpha}''$ in terms of $\vec{\alpha}_u$, $\vec{x}$, & derivatives

Since $\vec{\alpha}$ is assumed to be unit-speed,

$$\frac{d}{ds}\vec{T}(s) = \frac{d}{ds}\vec{\alpha}'(s) = \vec{\alpha}''(s) ,$$

so we'll need to work with $\vec{\alpha}''$.

By definition, $\vec{\alpha} = \underline{\vec{x} \circ \vec{\alpha}_u}$, so we use the <u>chain rule</u> to find $\vec{\alpha}'$:

$$\vec{\alpha}' = \sum_{i=1}^{2} \frac{\partial}{\partial u^i}(\vec{x})\Big|_{\vec{\alpha}_u(s)} \cdot \frac{d}{ds}\alpha_u^i(s)$$

$$= \sum_{i=1}^{2} \underbrace{\frac{d}{ds}\alpha_u^i(s)}_{\text{scalar}} \cdot \underbrace{\vec{x}_i(\vec{\alpha}_u(s))}_{\text{vector}}$$

$$(\vec{X}_i \circ \vec{\alpha}_u)(s) = \vec{X}_i(\alpha_u^1(s), \alpha_u^2(s))$$

$$\begin{tikzcd}
 & \boxed{\alpha_u^1} \arrow[dr, dash, "\frac{\partial \vec{X}_i}{\partial u^1}"] & \\
\boxed{s} \arrow[ur, dash, "\frac{d}{ds}\alpha_u^1"] \arrow[dr, dash, "\frac{d}{ds}\alpha_u^2"'] & & \boxed{\vec{X}_i \circ \vec{\alpha}_u} \\
 & \boxed{\alpha_u^2} \arrow[ur, dash, "\frac{\partial \vec{X}_i}{\partial u^2}"'] &
\end{tikzcd}$$

$$(\vec{X}_i \circ \vec{\alpha}_u)'(s) = \frac{\partial \vec{X}_i}{\partial u^1} \cdot \frac{d}{ds}\alpha_u^1 + \frac{\partial \vec{X}_i}{\partial u^2} \cdot \frac{d}{ds}\alpha_u^2$$

$$= \sum_{j=1}^{2} \vec{X}_{ij} \cdot \frac{d}{ds}\alpha_u^j$$

So $\vec{\alpha}'(s) = \sum_{i=1}^{2} \frac{d}{ds}\alpha_u^i(s) \cdot (\vec{X}_i \circ \vec{\alpha}_u)(s)$.

chain rule

To find $\vec{\alpha}''(s)$, we'll need the product and chain rules:

$$\vec{\alpha}''(s) = \sum_{i=1}^{2} \Big[\frac{d^2}{ds^2}\alpha_u^i(s) \cdot (\vec{X}_i \circ \vec{\alpha}_u)(s) + \frac{d}{ds}\alpha_u^i(s) \cdot \sum_{j=1}^{2} \frac{d}{ds}\alpha_u^j(s) \cdot \vec{X}_{ij}(\vec{\alpha}_u(s)) \Big]$$

$$= \sum_{i=1}^{2} \frac{d^2}{ds^2}\alpha_u^i(s) \cdot (\vec{X}_i \circ \vec{\alpha}_u)(s) + \sum_{i,j=1}^{2} \frac{d}{ds}\alpha_u^i(s) \cdot \frac{d}{ds}\alpha_u^j(s) \cdot (\vec{X}_{ij} \circ \vec{\alpha}_u)(s)$$

$$= \sum_{i=1}^{2} (\alpha_u^i)'' \cdot \vec{X}_i + \sum_{i,j=1}^{2} (\alpha_u^i)'(\alpha_u^j)'\, \vec{X}_{ij}.$$

$\vec{\alpha}_u(s) = (\alpha_u^1(s), \alpha_u^2(s)) \in \mathbb{R}^2$

Remember: we're trying to express κ_n & κ_g in terms of $\vec{\alpha}_u$, $\vec{x}$, and their derivatives. The formula

$$\vec{\alpha}'' = \underbrace{\sum_{i=1}^{2} (\alpha_u^i)'' \vec{x}_i}_{d\vec{x}\cdot\vec{\alpha}_u''} + \sum_{i,j=1}^{2} (\alpha_u^i)'(\alpha_u^j)' \vec{x}_{ij}$$

begins the process of breaking $\vec{\alpha}''$ into its "normal" and "geodesic" parts (but doesn't quite get there).

$$\vec{x}_{ij} = \frac{\partial^2}{\partial u_i \partial u_j} \vec{x} = \frac{\partial}{\partial u_j} \vec{x}_i$$