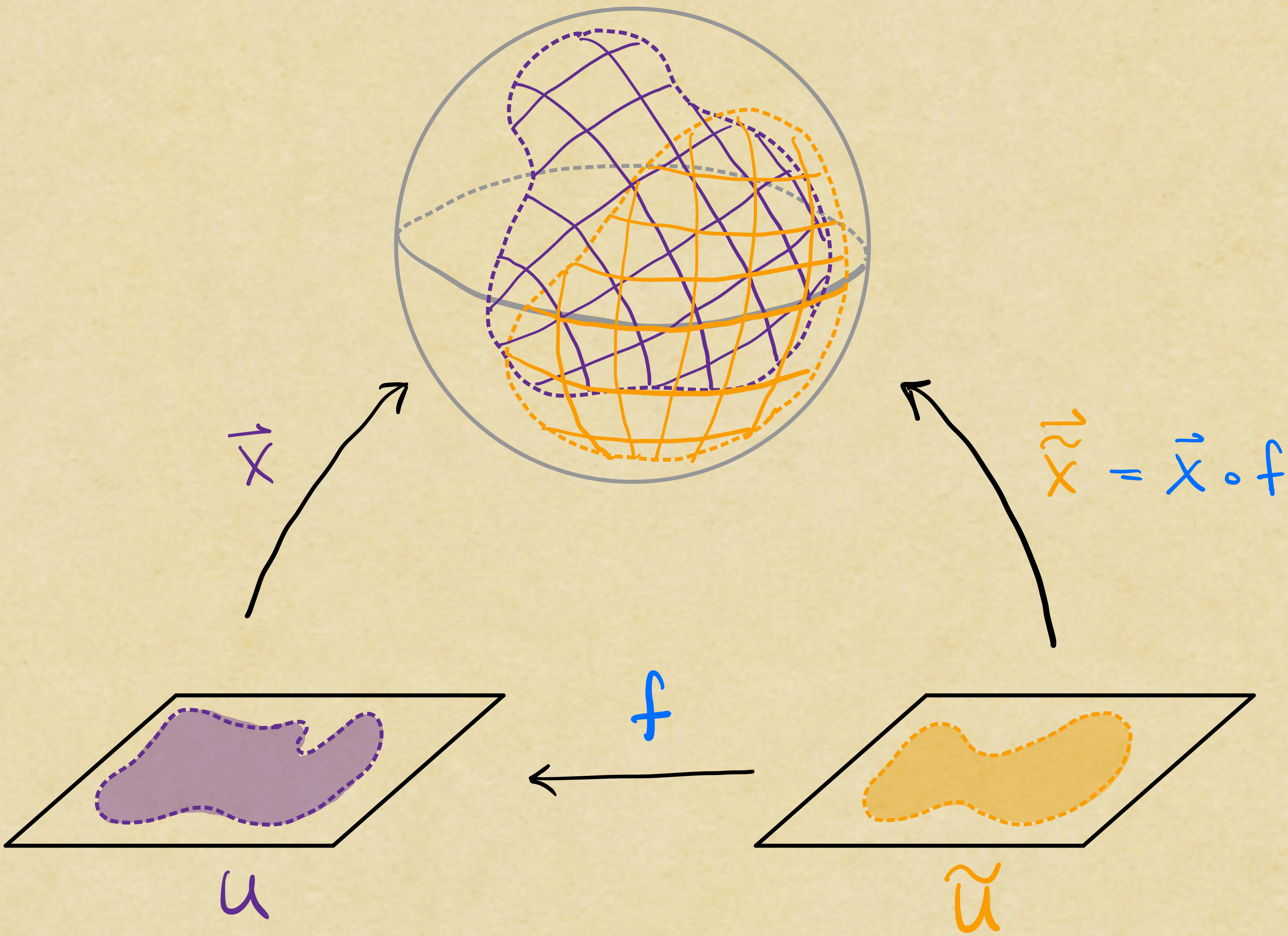
Math 4441 October 10, 2022

LAST TIME

We computed our most important geometric data for surfaces yet: the first fundamental form / metric tensor.

TODAY

How is the matrix of metric coefficients affected by coordinate transformations?



$\vec{x}$
$\tilde{\vec{x}} = \vec{x} \circ f$
f
U
$\tilde{U}$

Two reasons coordinate transformation laws matter

1. We want geometric invariants.
2. Plenty of things we might call "surfaces" can't be covered with a single simple surface, for topological reasons.

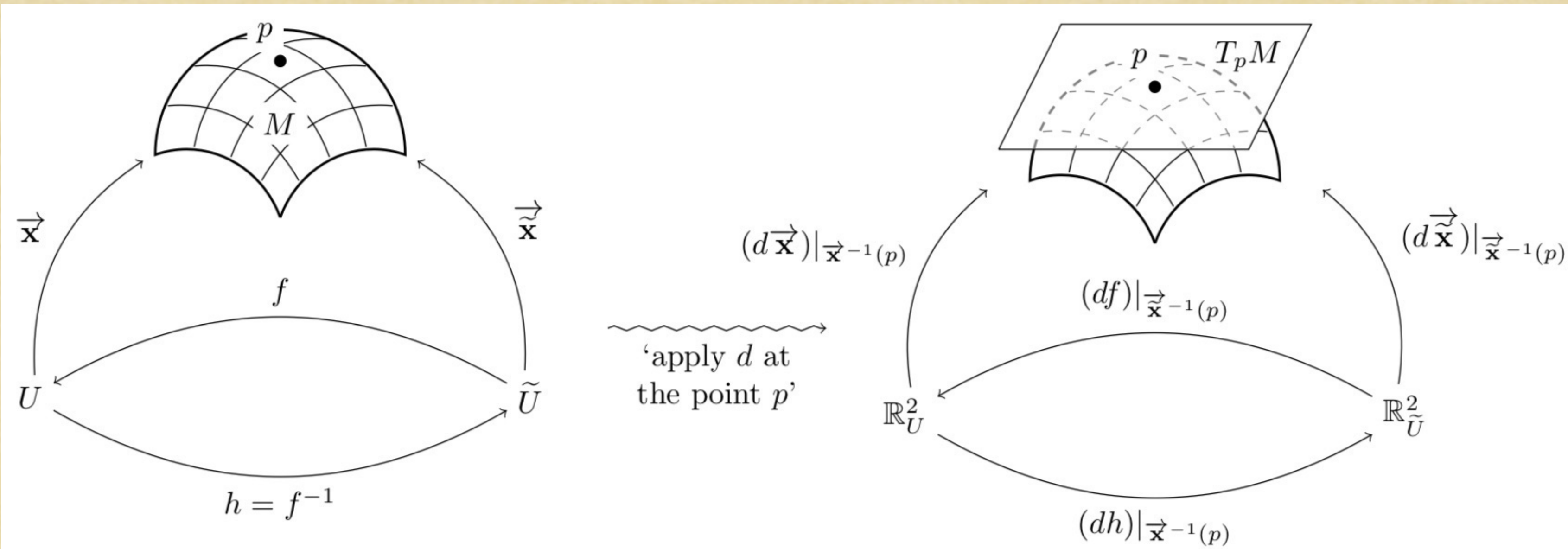
Given a simple surface $\vec{x}: U \to \mathbb{R}^3$ and a coordinate transformation $f: \tilde{U} \to U$, we define $\vec{\tilde{x}} = \underline{\vec{x} \circ f}$ and ask:

1. How are the bases $\underline{\{\vec{x}_1, \vec{x}_2\}}$ and $\underline{\{\vec{\tilde{x}}_1, \vec{\tilde{x}}_2\}}$ related? (Done)

2. When we write a tangent vector $\vec{X}$ in these bases, how are the coefficients $\underline{X^i}$ and $\underline{\tilde{X}^\alpha}$ related?

3. How are the metric coefficents $\underline{(g_{ij})}$ and $\underline{(\tilde{g}_{\alpha\beta})}$ related?

Our answers are essentially contained in this figure from the text:



$$\vec{\tilde{x}} = \vec{x} \circ f \quad \Rightarrow \quad d\vec{\tilde{x}} = d\vec{x} \circ df$$

① Getting $\{\vec{\tilde{x}}_1, \vec{\tilde{x}}_2\}$ from $\{\vec{x}_1, \vec{x}_2\}$.

This is a matter of writing out chain rule.

$$d\vec{\tilde{x}} = d\vec{x} \circ df \Rightarrow \begin{pmatrix} | & | \\ \vec{\tilde{x}}_1 & \vec{\tilde{x}}_2 \\ | & | \end{pmatrix} = \begin{pmatrix} | & | \\ \vec{x}_1 & \vec{x}_2 \\ | & | \end{pmatrix} \begin{pmatrix} \frac{\partial f^1}{\partial \tilde{u}^1} & \frac{\partial f^1}{\partial \tilde{u}^2} \\ \frac{\partial f^2}{\partial \tilde{u}^1} & \frac{\partial f^2}{\partial \tilde{u}^2} \end{pmatrix}$$

So $$\boxed{\vec{\tilde{x}}_j = \sum_{i=1}^{2} \frac{\partial f^i}{\partial \tilde{u}^j} \vec{x}_i}$$

About indices: i appears as both a subscript and a superscript $\Rightarrow$ we sum over i.

② Getting $\tilde{X}^\alpha, \tilde{X}^\beta$ from X^i, X^j.

Given $\vec{X} \in T_pM$, we can write

$$\vec{X} = \sum_{i=1}^{2} \underline{X^i}\, \vec{x}_i \quad \text{OR} \quad \vec{X} = \sum_{\alpha=1}^{2} \underline{\tilde{X}^\alpha}\, \vec{\tilde{x}}_\alpha .$$

So

$$\sum_{i=1}^{2} \underline{X^i}\, \vec{x}_i = \sum_{\alpha=1}^{2} \underline{\tilde{X}^\alpha}\, \vec{\tilde{x}}_\alpha$$

$$= \sum_{\alpha=1}^{2} \tilde{X}^\alpha \left(\sum_{i=1}^{2} \frac{\partial f^i}{\partial \tilde{u}^\alpha} \vec{x}_i \right) \quad (\text{from } ①)$$

$$= \sum_{i=1}^{2} \left(\sum_{\alpha=1}^{2} \tilde{X}^\alpha \cdot \frac{\partial f^i}{\partial \tilde{u}^\alpha} \right) \vec{x}_i$$

Upshot: $X^i = \sum_{\alpha=1}^{2} \tilde{X}^\alpha \cdot \frac{\partial f^i}{\partial \tilde{u}^\alpha}$.

② Getting $\tilde{X}^\alpha, \tilde{X}^\beta$ from X^i, X^j.

But we wanted to go the other way. No problem.

Let's write $h = f^{-1} : U \longrightarrow \tilde{U}$. Then, just as we showed

$$\boxed{X^i = \sum_{\alpha=1}^{2} \frac{\partial f^i}{\partial \tilde{u}^\alpha} \tilde{X}^\alpha}$$

we may find that

$$\boxed{\tilde{X}^\alpha = \sum_{i=1}^{2} \frac{\partial h^\alpha}{\partial u^i} X^i \quad .}$$

Only catch for both formulas: pay attention to where functions are evaluated.

③ Getting $(\tilde{g}_{\alpha\beta})$ from (g_{ij}).

Once again, we'll need our formulas from ①:

$$\vec{\tilde{X}}_\alpha = \sum_{i=1}^{2} \frac{\partial f^i}{\partial \tilde{u}^\alpha} \vec{X}_i \quad ; \quad \vec{\tilde{X}}_\beta = \sum_{j=1}^{2} \frac{\partial f^j}{\partial \tilde{u}^\beta} \vec{X}_j .$$

Then

$$\tilde{g}_{\alpha\beta} = \langle \vec{\tilde{X}}_\alpha, \vec{\tilde{X}}_\beta \rangle$$

$$= \left\langle \sum_{i=1}^{2} \frac{\partial f^i}{\partial \tilde{u}^\alpha} \vec{X}_i , \sum_{j=1}^{2} \frac{\partial f^j}{\partial \tilde{u}^\beta} \vec{X}_j \right\rangle$$

$$= \sum_{i,j=1}^{2} \left\langle \frac{\partial f^i}{\partial \tilde{u}^\alpha} \vec{X}_i , \frac{\partial f^j}{\partial \tilde{u}^\beta} \vec{X}_j \right\rangle$$

$$= \sum_{i,j=1}^{2} \frac{\partial f^i}{\partial \tilde{u}^\alpha} \cdot \frac{\partial f^j}{\partial \tilde{u}^\beta} \overbrace{\langle \vec{X}_i , \vec{X}_j \rangle}^{g_{ij}}$$

So we find that

$(\tilde{g}_{\alpha\beta}) = (df)^T (g_{ij})\, df$

$$\boxed{\tilde{g}_{\alpha\beta} = \sum_{i,j=1}^{2} \frac{\partial f^i}{\partial \tilde{u}^\alpha} \cdot \frac{\partial f^j}{\partial \tilde{u}^\beta} \cdot g_{ij}}$$

Similarly,

$$\boxed{g_{ij} = \sum_{\alpha,\beta=1}^{2} \frac{\partial h^\alpha}{\partial u^i} \cdot \frac{\partial h^\beta}{\partial u^j}\, \tilde{g}_{\alpha\beta}},$$

$(g_{ij}) = (dh)^T (\tilde{g}_{\alpha\beta})\, dh$

where $h = f^{-1}$.