Math 4441 November 9, 2022

LAST TIME

We defined the Weingarten map, which measures the shape of a surface via directional derivatives of the unit normal vector (or Gauss map).

TODAY

We'll discuss the linearity of the Weingarten map, and use the function $\vec{x} : U \to \mathbb{R}^3$ to obtain a matrix representation of $\mathcal{L}$.

Maybe: continue thinking about that minus sign.

Recall:

Def The Gauss map for an oriented surface $S \subset \mathbb{R}^3$ is the map $\nu : S \to S^2$ defined by $\nu(p) :=$ unit surface normal @ p

i.e., $\vec{n} = \nu \circ \vec{x}$

Def The Weingarten map (or shape operator) of a surface S at a point $p \in S$ is a linear map

$$\mathcal{L} = \mathcal{L}_p : T_pS \to \mathbb{R}^3$$

defined by

$$\mathcal{L}(\vec{v}) := -D_{\vec{v}}\,\nu \, .$$

What does $\mathcal{L}$ do to $\vec{x}_i$?

$$\mathcal{L}(\vec{x}_i) := -D_{\vec{x}_i}\nu = -(\nu\circ\vec{\alpha}_i)'(0),$$

where $\vec{\alpha}_1(t) = \vec{x}(u_0^1+t,\ u_0^2)$ $\quad \vec{\alpha}_1(0) = \vec{x}(u_0^1, u_0^2) = p$

& $\vec{\alpha}_2(t) = \vec{x}(u_0^1,\ u_0^2+t)$. $\quad \vec{\alpha}_1'(0) = \vec{x}_1(u_0^1, u_0^2)$

Now $(\nu\circ\vec{\alpha}_1)(t) = (\nu\circ\vec{x})(u_0^1+t, u_0^2)$

$$= \vec{n}(u_0^1+t, u_0^2)$$

So $(\nu\circ\vec{\alpha}_1)'(t) = \frac{d}{dt}(u_0^1+t)\cdot\frac{\partial\vec{n}}{\partial u^1} + \cancel{\frac{d}{dt}(u_0^2)}\cdot\frac{\partial\vec{n}}{\partial u^2}$

$$= \frac{\partial\vec{n}}{\partial u^1}.$$

($i=2$ case similar)

Then $\boxed{\mathcal{L}(\vec{x}_i) = -\frac{\partial\vec{n}}{\partial u^i} =: -\vec{n}_i}$

Proposition. The Weingarten map

$$\mathcal{L} : T_p\, \mathrm{im}(\vec{x}) \longrightarrow \mathbb{R}^3$$

is linear, with image contained in $T_p\, \mathrm{im}(\vec{x})$.

(Proof.) The linearity of the Weingarten map will follow from the linearity of directional derivatives w.r.t. the direction vector.

We need to show that, for any $a, b \in \mathbb{R}$ and $\vec{v}, \vec{w} \in T_p\, \mathrm{im}(\vec{x})$,

$$\mathcal{L}(a\vec{v} + b\vec{w}) = a\mathcal{L}(\vec{v}) + b\,\mathcal{L}(\vec{w}).$$

Indeed,

$$\begin{aligned}\mathcal{L}(a\vec{v}+b\vec{w}) &= -D_{a\vec{v}+b\vec{w}}\nu \\ &= -aD_{\vec{v}}\nu - bD_{\vec{w}}\nu \qquad \text{(activity 10)} \\ &= a\mathcal{L}(\vec{v}) + b\mathcal{L}(\vec{w}).\end{aligned}$$

So $\mathcal{L}$ is linear. We can now address the image more carefully. Since $\{\vec{x}_1, \vec{x}_2\}$ is a basis for the domain $T_p\, \mathrm{im}(\vec{x})$ of $\mathcal{L}$, $\{\mathcal{L}(\vec{x}_1), \mathcal{L}(\vec{x}_2)\} = \{-\vec{n}_1, -\vec{n}_2\}$ will span the image.

Let's investigate these vectors.

Since $\|\vec{n}\| \equiv 1$, $\langle \vec{n}, \vec{n} \rangle \equiv 1$, so

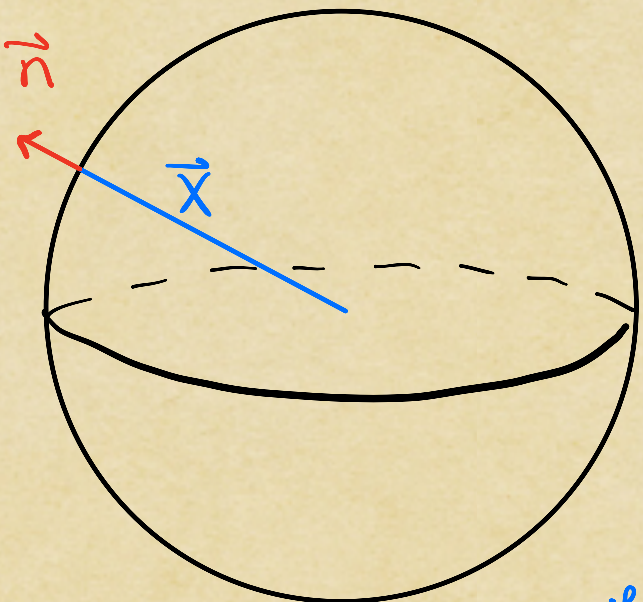
$$\langle \vec{n}_i, \vec{n} \rangle + \langle \vec{n}, \vec{n}_i \rangle \equiv 0$$

That is, $\vec{n}_i \perp \vec{n}$. So the image of $\mathcal{L}$ is $\underline{\perp \vec{n}}$, and thus lies in $T_p \operatorname{im}(\vec{x})$. $\diamond$

Since $\mathcal{L} : T_p \operatorname{im}(\vec{x}) \longrightarrow T_p \operatorname{im}(\vec{x})$ is linear and we have a preferred basis $\{\vec{x}_1, \vec{x}_2\}$, we can get a <u>matrix representation</u> for $\mathcal{L}$:

$$[\mathcal{L}]_{\{\vec{x}_1, \vec{x}_2\}} =: \begin{pmatrix} L_1^1 & L_2^1 \\ L_1^2 & L_2^2 \end{pmatrix}.$$

Example Consider *any* simple surface $\vec{x}: U \to \mathbb{R}^3$ whose image is the sphere of radius R centered at the origin.



Then $\vec{n} = \frac{1}{R}\vec{x}$

Thus
$$\begin{aligned}\mathcal{L}(\vec{x}_i) &= -\vec{n}_i \\ &= -\frac{\partial}{\partial u^i}\left(\frac{1}{R}\vec{x}\right) \\ &= -\frac{1}{R}\vec{x}_i\end{aligned}$$

That is,

$$\mathcal{L}(\vec{x}_1) = -\frac{1}{R}\cdot\vec{x}_1 + 0\cdot\vec{x}_2$$
$$\mathcal{L}(\vec{x}_2) = 0\cdot\vec{x}_1 + -\frac{1}{R}\cdot\vec{x}_2$$

$$\Rightarrow \left(L_i^j\right) = \begin{array}{c c} \mathcal{L}(\vec{x}_1) & \mathcal{L}(\vec{x}_2) \\ \multicolumn{2}{c}{\begin{pmatrix} -1/R & 0 \\ 0 & -1/R \end{pmatrix}} \end{array}$$

Remarks

- This direct approach to computing (L^j_i) gets messy quickly; we'll see a better way soon.

- Remember what $\mathcal{L}$ measures: how quickly $\underline{v}$ changes as we move in the given direction.

- Unlike (L_{ij}) (don't get these confused!), (L^j_i) need not be symmetric.

We can now write some version of the Frenet-Serret equations for the surface basis $\{\vec{X}_1, \vec{X}_2, \vec{n}\}$:

$$\vec{X}_{ij} = L_{ij}\,\vec{n} + \sum_{k=1}^{2} \Gamma_{ij}^{k}\,\vec{X}_k$$

$$\vec{n}_k = -\sum_{i=1}^{2} L_k^i\,\vec{X}_i$$

I don't expect to use these much, but we'll study relationships among L_{ij}, Γ_{ij}^k, & L_i^j.