Math 4441 November 7, 2022

LAST TIME

We defined differentiation for the setup

$f : \underline{S \to \mathbb{R}^n}$, $p \in \underline{S}$, $\vec{v} \in \underline{T_pS}$.

The <u>directional derivative</u> $D_{\vec{v}} f$ tells us how f changes when we <u>leave p</u> with <u>velocity vector $\vec{v}$</u>.

TODAY

We'll attempt to quantify the <u>shape</u> of a surface by imitating our def'n of the <u>curvature</u> of a curve.

This is a course in the differential geometry of curves and surfaces, so we study curves and surfaces by differentiating.

1. Curves in $\mathbb{R}^2$ or $\mathbb{R}^3$.
   Use the geometry of ambient $\mathbb{R}^n$ to study $\vec{\alpha}$.
2. Curves in a surface.
   Study $\vec{\alpha}$ via the "intrinsic" geometry of the surface.
3. Surfaces in $\mathbb{R}^3$
   Use the geometry of ambient $\mathbb{R}^3$ to study $\vec{x}$.

So we're going to repeat the curves section, now with surfaces.

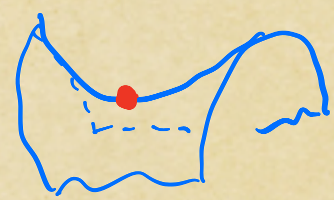
But! Surfaces $\neq$ curves

What are some important differences? (Especially for differential geometers.)

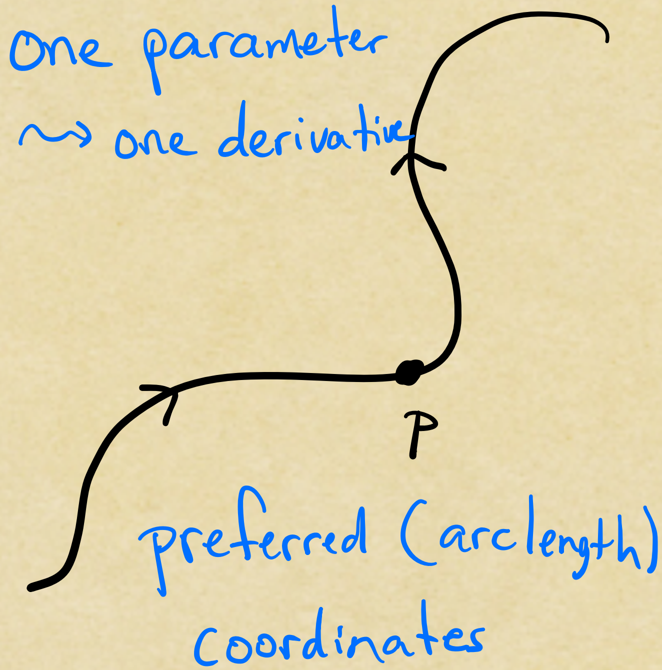
Surfaces: 2 parameters (metric tensor)
curves : 1 parameter (arc length)

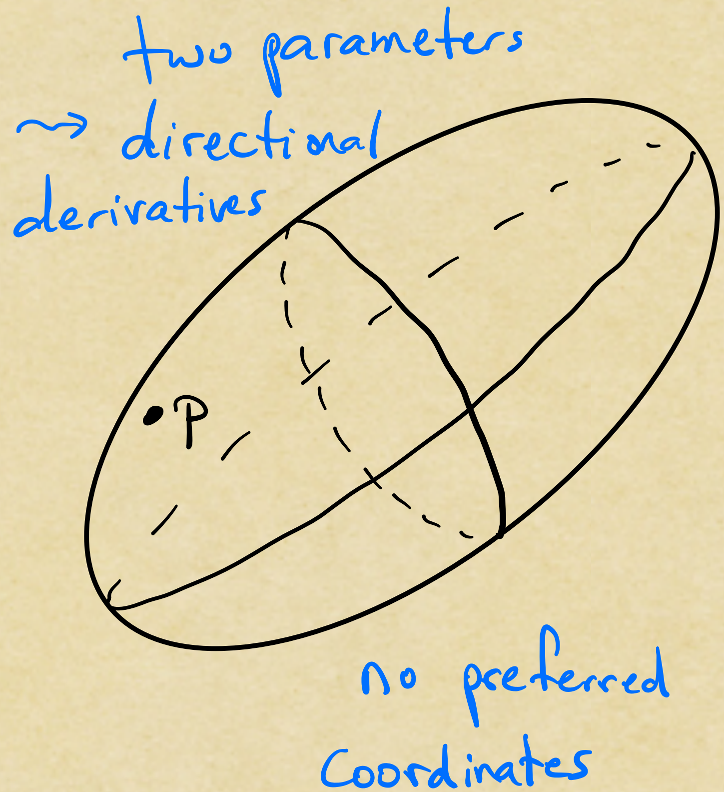
we need directional derivatives

Convexity is way different



Friday's activity addressed one difference that must be dealt with immediately: differentiation.





Directional derivatives

Recall that if we have $f : \operatorname{im} \vec{x} \to \mathbb{R}^n$,
$p \in \operatorname{im} \vec{x}$, & $\vec{v} \in T_p \operatorname{im} \vec{x}$, then

$$D_{\vec{v}} f := (f \circ \vec{\alpha})'(0),$$

where $\vec{\alpha}$ satisfies $\underline{\vec{\alpha}(0) = p}$ and $\underline{\vec{\alpha}'(0) = \vec{v}}$.

Facts:

1. $D_{a\vec{v} + b\vec{w}} f = a D_{\vec{v}} f + b D_{\vec{w}} f$
2. $D_{\vec{v}}(af + bg) = a D_{\vec{v}} f + b D_{\vec{v}} g$
3. $D_{\vec{v}}(\langle \vec{X}, \vec{Y} \rangle) = \langle D_{\vec{v}} \vec{X}, \vec{Y} \rangle + \langle \vec{X}, D_{\vec{v}} \vec{Y} \rangle$

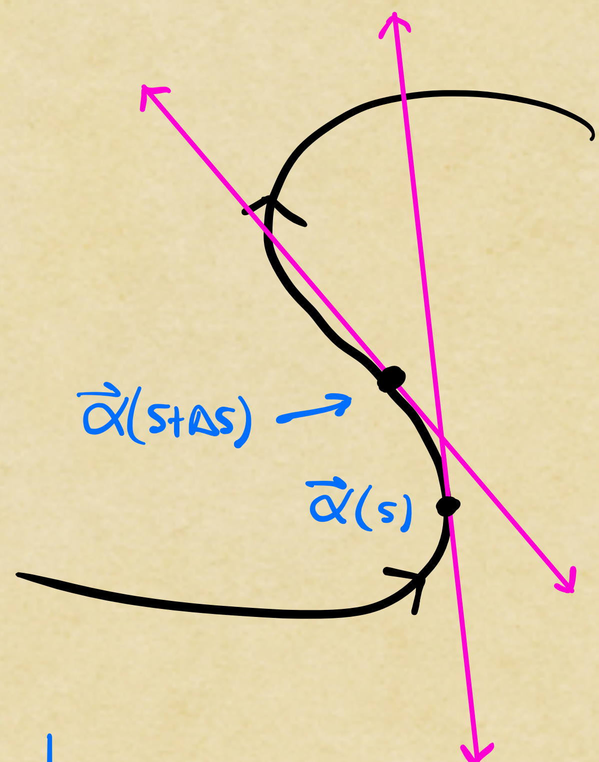
Our book writes $\vec{v} f$ where we write $D_{\vec{v}} f$!!

Towards curvature

For curves in $\mathbb{R}^3$:

$$\kappa = \left\langle \frac{d}{ds}\vec{T}, \vec{N} \right\rangle$$

We think about $\frac{d}{ds}\vec{T}$ because it tells us how the tangent line changes as we move along the curve with unit speed.

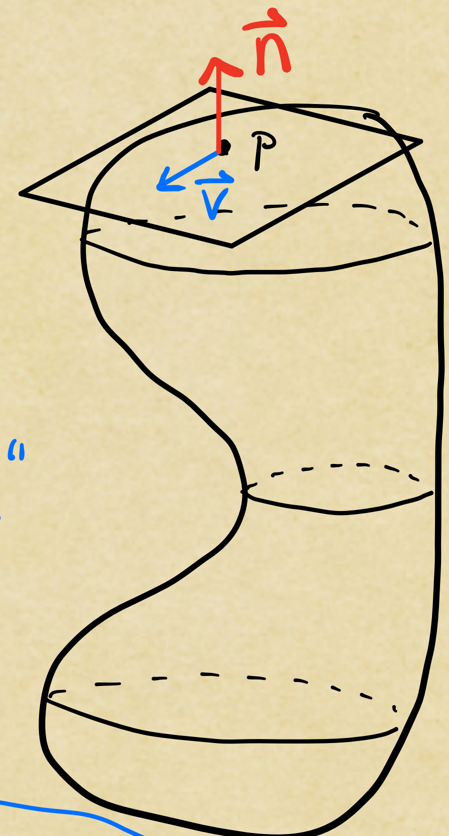


Towards curvature

For ~~curves~~ surfaces in $\mathbb{R}^3$:

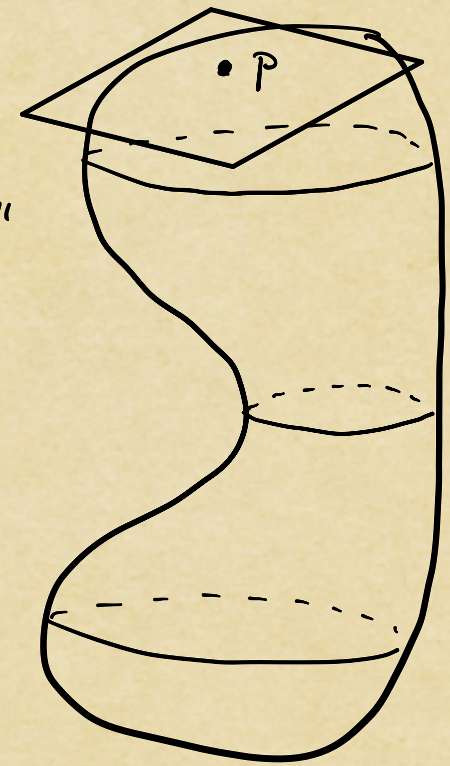
$\kappa =$ TBD

We think about $\cancel{\frac{d}{ds}\vec{T}}$ "$D_{\vec{v}}\vec{n}$"
because it tells us
how ~~the tangent line~~ the tangent plane
changes as we ~~move along~~
~~the curve with unit speed.~~
travel with velocity vector $\vec{v} \in T_pS$



Towards curvature

For surfaces in $\mathbb{R}^3$:



① "How does tangent plane change?"

= "How does $\vec{n}$ change?"

② "How does <whatever> change?"

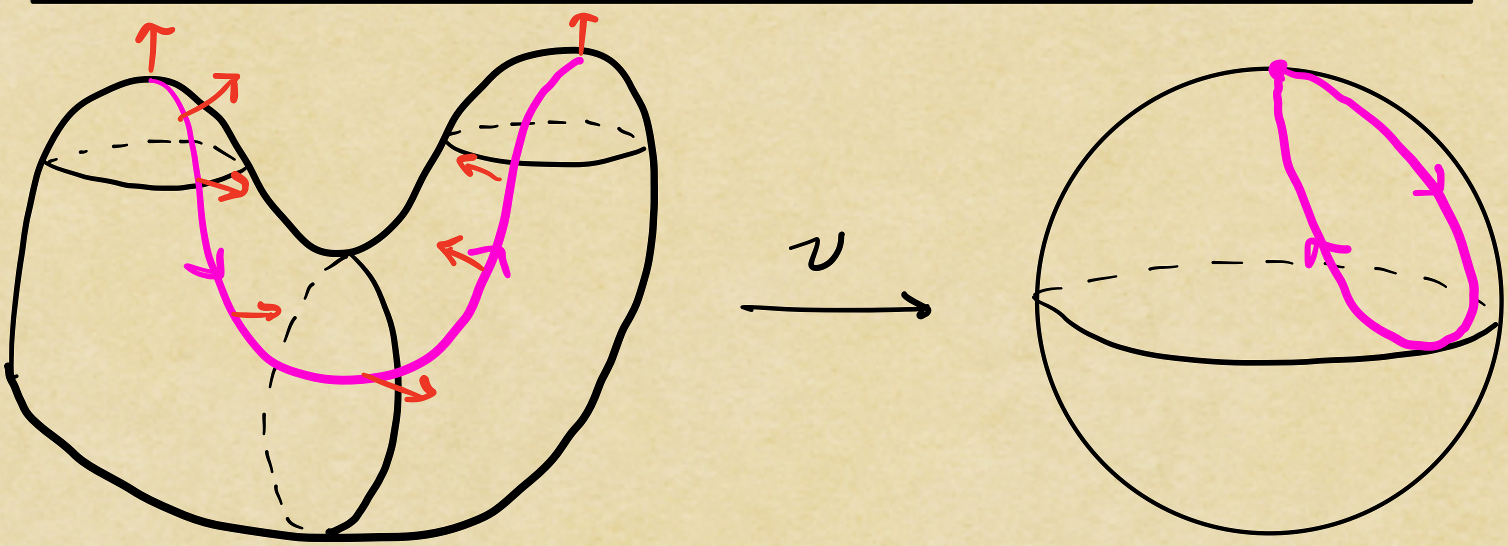
really requires directional derivatives.

Upshot: We'll take a step towards curvature by computing the derivative of $\vec{n}$ in the direction $\vec{v}$.

Making this precise

Def The Gauss map for an oriented surface $S \subset \mathbb{R}^3$ is the map $\nu : \underline{S} \longrightarrow \underline{S^2}$ defined by $\nu(p) :=$ <u>the unit surface normal @ p</u>

i.e., $\vec{n} = \underline{\nu \circ \vec{x}}$



Making this precise

Def The Weingarten map (or shape operator) of a surface S at a point $p \in S$ is a linear map

$$\mathcal{L} = \mathcal{L}_p : T_pS \longrightarrow \mathbb{R}^3$$

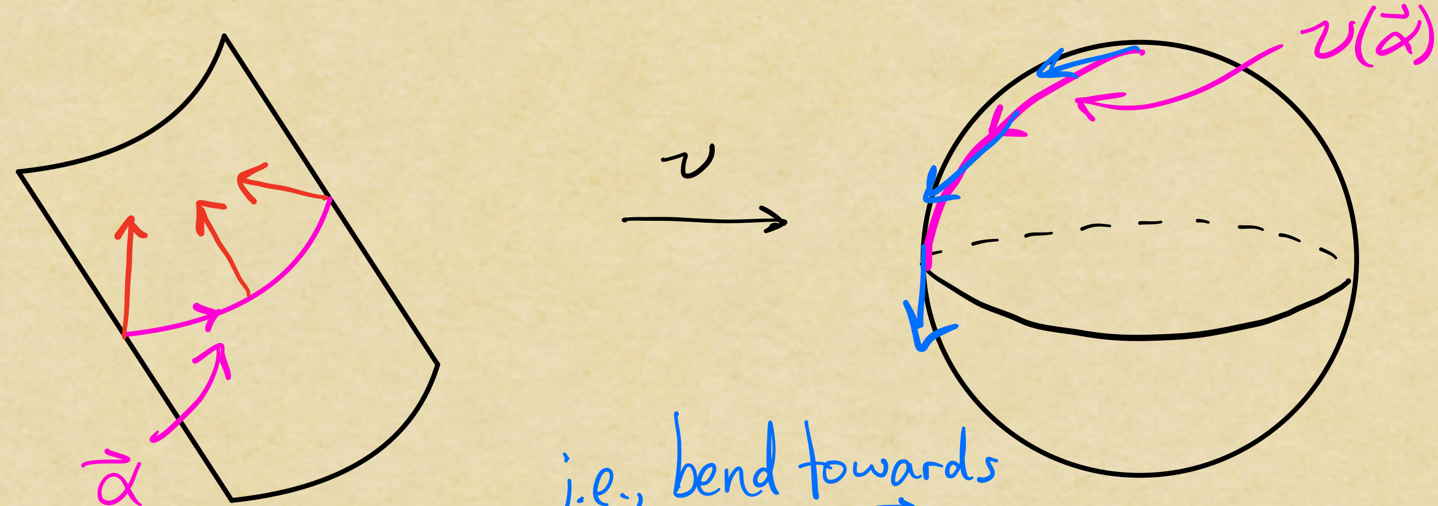
defined by

$$\mathcal{L}(\vec{v}) := -D_{\vec{v}}\nu \; .$$

Remarks

1. $\mathcal{L}$ is "the" derivative of ν at p — it's a linear map
2. We'll explain the minus sign shortly.

Why the minus sign? (take #1)



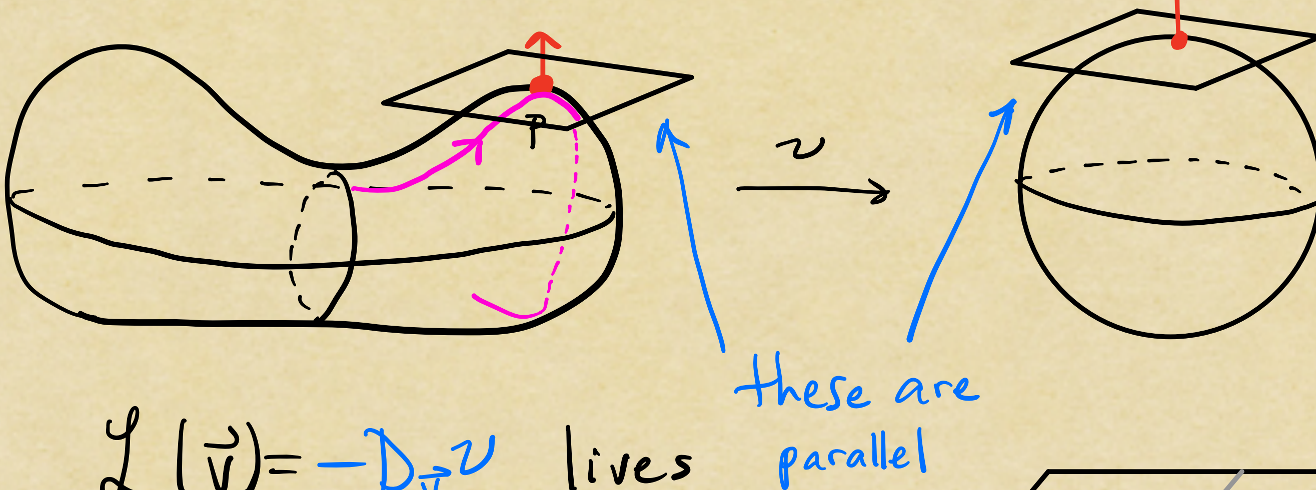
i.e., bend towards $\vec{n}$

Say we're following a path where the surface seems to go up — then $\vec{n}$ would appear to be moving backwards. i.e., $D_{\vec{\alpha}'}\nu$ would point against $\vec{\alpha}'$, so $\langle \vec{\alpha}', D_{\vec{\alpha}'}\nu \rangle$ < 0.

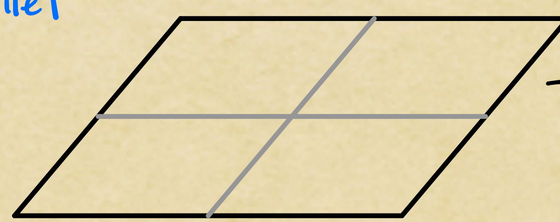
But we'd prefer "up" corresponding to >0.

The range of $\mathcal{L}$ (time permitting)

it did not



$\mathcal{L}(\vec{v}) = \underline{-D_{\vec{v}}\nu}$ lives in $\underline{T_{\nu(p)} S^2}$



$T_{\nu(p)} S^2$

The unit sphere S^2 has a nice property: the normal vector at any point is just the point itself.

So the normal vector to $T_{\nu(p)} S^2$ is $\underline{\nu(p) = \vec{n}(p)}$.

But this is the normal vector to $T_p S$!

The range of $\mathcal{L}$ (time permitting)

Upshot: We think of $\mathcal{L} : T_pS \to \mathbb{R}^3$ as a map

$$\mathcal{L} : \underline{T_pS \to T_pS}.$$

This allows us to represent $\mathcal{L}$ with a $\underline{2\times 2}$ $\underline{\text{matrix}}$ (once we know it's $\underline{\text{linear}}$)