Math 4441 November 30, 2022

LAST TIME

We interpreted the Gaussian curvature as a stretch factor for the Gauss map. This understanding of K is very extrinsic.

TODAY

We'll find relations between the first and second fundamental forms, eventually proving that Gaussian curvature is an intrinsic quantity!

We saw in Activity 11 that the plane and unit cylinder both have first fundamental form

$$(g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

but that their second fundamental forms are

$$(L_{ij}) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{\&} \quad (L_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

respectively.

<u>Upshot</u>: The SFF <u>cannot be computed</u> from the FFF — so it's not intrinsic.

But maybe some relations exist...

Because the Christoffel symbols Γ_{ij}^{k} are intrinsic, we think of them as providing "first order" information.

The key link between first-order and second-order information is then the equation

$$\vec{x}_{ij} = L_{ij}\vec{n} + \sum_{k=1}^{2} \Gamma_{ij}^{k} \vec{x}_k \qquad (\star)$$

Today we'll exploit this link to deduce remarkable theorem.

We'll take advantage of (★) by differentiating:

$$\frac{\partial}{\partial u^k}\vec{x}_{ij} = \vec{x}_{ijk} = \frac{\partial}{\partial u^j}\vec{x}_{ik}$$

For concreteness, let's just consider $i=j=1$ and $k=2$. Other cases are similar.

$$\vec{x}_{11} = L_{11}\vec{n} + \Gamma_{11}^{1}\vec{x}_1 + \Gamma_{11}^{2}\vec{x}_2$$

$$\Rightarrow \vec{x}_{112} = (L_{11})_2\vec{n} + L_{11}\vec{n}_2 + (\Gamma_{11}^{1})_2\vec{x}_1 + \Gamma_{11}^{1}\vec{x}_{12} + (\Gamma_{11}^{2})_2\vec{x}_2 + \Gamma_{11}^{2}\vec{x}_{22}$$

$$= (L_{11})_2\vec{n} + L_{11}\left(-L_2^1\vec{x}_1 - L_2^2\vec{x}_2\right) + (\Gamma_{11}^{1})_2\vec{x}_1 + (\Gamma_{11}^{2})_2\vec{x}_2 + \Gamma_{11}^{1}\left(L_{12}\vec{n} + \Gamma_{12}^{1}\vec{x}_1 + \Gamma_{12}^{2}\vec{x}_2\right) + \Gamma_{11}^{2}\left(L_{22}\vec{n} + \Gamma_{22}^{1}\vec{x}_1 + \Gamma_{22}^{2}\vec{x}_2\right)$$

Gathering like terms,

$$\begin{aligned}\vec{X}_{112} = {} & \left[(L_{11})_2 + \Gamma^1_{11} L_{12} + \Gamma^2_{11} L_{22}\right]\vec{n} \\ & + \left[\Gamma^1_{11}\Gamma^1_{12} + \Gamma^2_{11}\Gamma^1_{22} + (\Gamma^1_{11})_2 - L_{11}L^1_2\right]\vec{x}_1 \\ & + \left[\Gamma^1_{11}\Gamma^2_{12} + \Gamma^2_{11}\Gamma^2_{22} + (\Gamma^2_{11})_2 - L_{11}L^2_2\right]\vec{x}_2 .\end{aligned}$$

Similarly, differentiating $\vec{X}_{12} = L_{12}\vec{n} + \Gamma^1_{12}\vec{x}_1 + \Gamma^2_{12}\vec{x}_2$ w.r.t. $\underline{u^1}$ gives

$$\begin{aligned}\vec{X}_{121} = {} & \left[(L_{12})_1 + \Gamma^1_{12} L_{11} + \Gamma^2_{12} L_{21}\right]\vec{n} \\ & + \left[\Gamma^1_{12}\Gamma^1_{11} + \Gamma^2_{12}\Gamma^1_{21} + (\Gamma^1_{12})_2 - L_{12}L^1_1\right]\vec{x}_1 \\ & + \left[\Gamma^1_{12}\Gamma^2_{11} + \Gamma^2_{12}\Gamma^2_{21} + (\Gamma^2_{12})_1 - L_{12}L^2_1\right]\vec{x}_2\end{aligned}$$

Because $\underline{\vec{X}_{121} = \vec{X}_{112}}$, we can equate, say, $\vec{x}_1$ - components:

$$\cancel{\Gamma^1_{11}\Gamma^1_{12}} + \Gamma^2_{11}\Gamma^1_{22} + (\Gamma^1_{11})_2 - L_{11}L^1_2 = \cancel{\Gamma^1_{12}\Gamma^1_{11}} + \Gamma^2_{12}\Gamma^1_{21} + (\Gamma^1_{12})_2 - L_{12}L^1_1$$

This looks wild, but from

$$\Gamma_{11}^{2}\Gamma_{22}^{1} + (\Gamma_{11}^{1})_2 - L_{11}L_2^1 = \Gamma_{12}^{2}\Gamma_{21}^{1} + (\Gamma_{12}^{1})_1 - L_{12}L_1^1$$

we should be able to separate out first- and second-order information.

Namely,

$$\underbrace{L_{11}L_2^1 - L_{12}L_1^1}_{\text{"second-order"}} = \underbrace{\Gamma_{11}^{2}\Gamma_{22}^{1} + (\Gamma_{11}^{1})_2 - \Gamma_{12}^{2}\Gamma_{21}^{1} - (\Gamma_{12}^{1})_1}_{\text{"first-order"}}.$$

Maybe this is still unimpressive, so let's think about the LHS a bit. Recall that $(L_{ij}) = (g_{ij})(L_j^i)$.

$$\begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}\begin{pmatrix} L_1^1 & L_2^1 \\ L_1^2 & L_2^2 \end{pmatrix} \Rightarrow \begin{aligned} L_{11} &= g_{11}L_1^1 + g_{12}L_1^2 \\ L_{12} &= g_{11}L_2^1 + g_{12}L_2^2 \end{aligned}$$

Okay, so

$$L_{11}L^1_2 - L_{12}L^1_1 = \Gamma^2_{11}\Gamma^1_{22} + (\Gamma^1_{11})_2 - \Gamma^2_{12}\Gamma^1_{21} - (\Gamma^1_{12})_1 ,$$

$$L_{11} = g_{11}L^1_1 + g_{12}L^2_1 , \quad \& \quad L_{12} = g_{11}L^1_2 + g_{12}L^2_2 .$$

Then

$$\begin{aligned} L_{11}L^1_2 - L_{12}L^1_1 &= (g_{11}L^1_1 + g_{12}L^2_1)L^1_2 - (g_{11}L^1_2 + g_{12}L^2_2)L^1_1 \\ &= \cancel{g_{11}L^1_1L^1_2} + g_{12}L^2_1L^1_2 \cancel{- g_{11}L^1_2L^1_1} - g_{12}L^2_2L^1_1 \\ &= g_{12}(L^2_1L^1_2 - L^1_1L^2_2) \\ &= -g_{12}\cdot\det(L^i_j) \quad = -g_{12}\cdot K \end{aligned}$$

So at last we have

$$\boxed{-g_{12}\cdot K = \Gamma^2_{11}\Gamma^1_{22} + (\Gamma^1_{11})_2 - \Gamma^2_{12}\Gamma^1_{21} - (\Gamma^1_{12})_1}$$

We derived this equation by comparing $\vec{x}_1$-components in the equation $(\vec{x}_{11})_2 = (\vec{x}_{12})_1$. By comparing the $\vec{x}_2$-components, we obtain

$$\boxed{g_{11}K = \Gamma_{11}^1\Gamma_{12}^2 + \Gamma_{11}^2\Gamma_{22}^2 + (\Gamma_{11}^2)_2 - \Gamma_{12}^1\Gamma_{11}^2 - \Gamma_{12}^2\Gamma_{21}^2 - (\Gamma_{12}^2)_1},$$

and by comparing $\vec{x}_1$- and $\vec{x}_2$- components of $(\vec{x}_{22})_1 = (\vec{x}_{12})_2$, we obtain

$$\boxed{g_{22}K = \Gamma_{22}^1\Gamma_{11}^1 + \Gamma_{22}^2\Gamma_{12}^1 + (\Gamma_{22}^1)_1 - \Gamma_{12}^1\Gamma_{21}^1 - \Gamma_{12}^2\Gamma_{22}^1 - (\Gamma_{12}^1)_2}$$

and

$$\boxed{-g_{12}K = \Gamma_{11}^2\Gamma_{22}^1 - (\Gamma_{12}^2)_2 - \Gamma_{12}^2\Gamma_{21}^1 + (\Gamma_{22}^2)_1}.$$

These four are called the Gauss equations.

__Corollary__ (Theorema Egregium)

The Gaussian curvature K is intrinsic.

(Proof.) At least one of g_{11}, g_{12}, g_{22} must be nonzero. Solve the corresponding equation for K. $\diamond$

__Corollary__ We know how to eat pizza.

Other important equations

Notice that we only ever considered the tangential components of $\vec{X}_{ijk}$. By considering normal components we obtain the Codazzi-Mainardi equations:

$$\boxed{\begin{aligned}(L_{11})_2-(L_{12})_1&=L_{11}\Gamma^1_{12}+L_{12}(\Gamma^2_{12}-\Gamma^1_{11})-L_{22}\Gamma^2_{11}\\ \text{\&}\ (L_{12})_2-(L_{22})_1&=L_{11}\Gamma^1_{22}+L_{12}(\Gamma^2_{22}-\Gamma^1_{12})-L_{22}\Gamma^2_{12}.\end{aligned}}$$

These also give useful second-order information from the first fundamental form. (In fact, no further relations between FFF & SFF exist.)