Math 4441 November 28, 2022

LAST TIME

From the Weingarten map $\mathcal{L}$ we extracted the

- principal curvatures — its eigenvalues;
- Gaussian curvature — its determinant
- mean curvature — half its trace.

TODAY

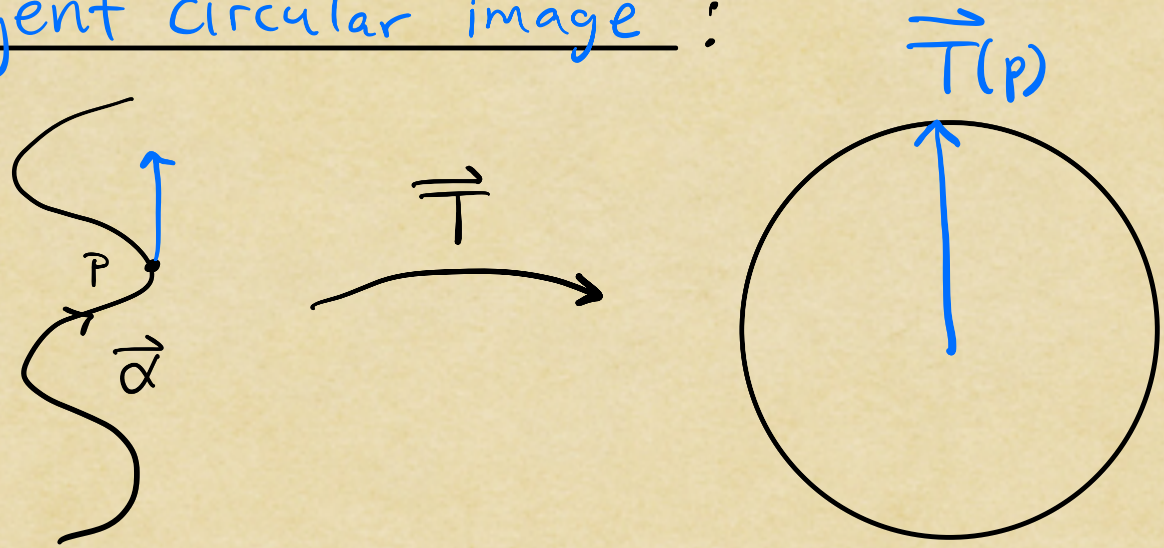
We want to see the Gaussian curvature as a "stretch factor" for the Gauss map.

Plan

1. Interpret planar curvature as a stretch factor.
2. Define surface integration.
3. Interpret Gaussian curvature as a stretch factor.

Planar curvature as a stretch factor

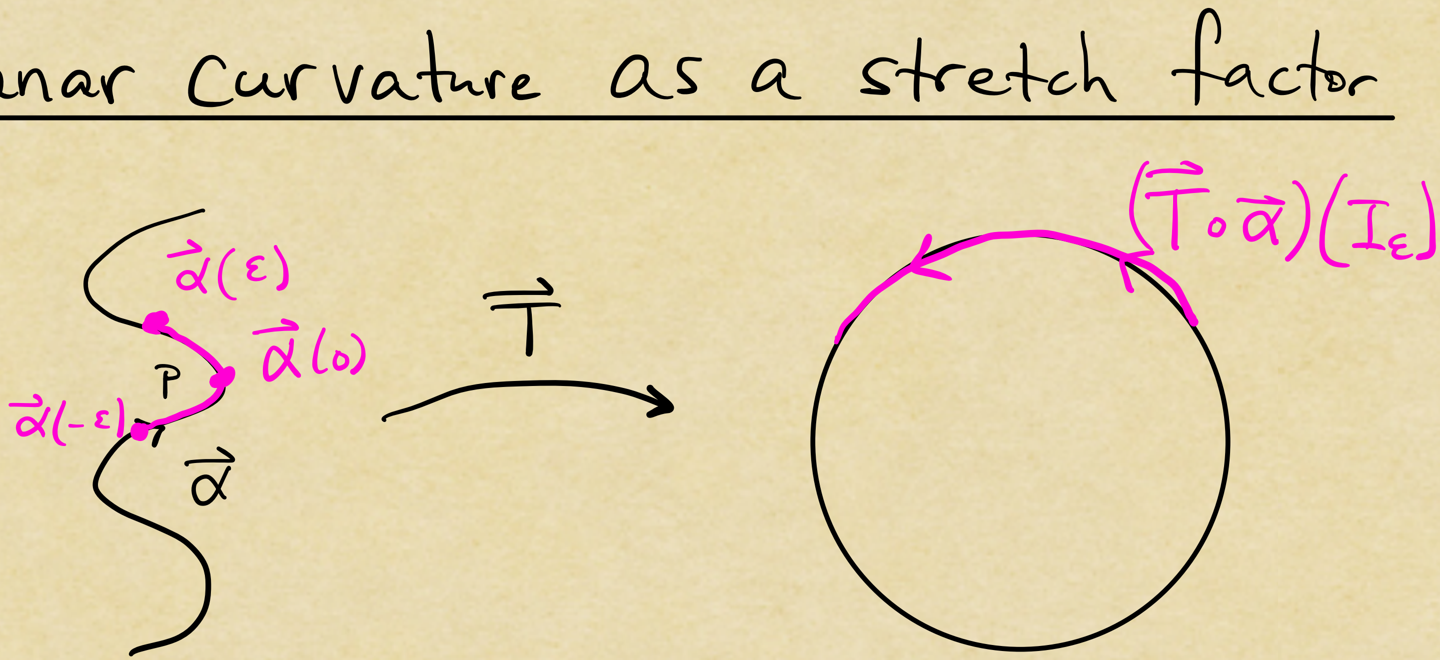
Given a planar, unit-speed $\vec{\alpha}$, consider the <u>tangent circular image</u>:



Let's consider the behavior near $p = \underline{\vec{\alpha}(0)}$.

In particular, we want to know how $\underline{\vec{T} \circ \vec{\alpha}}$ behaves on the interval $I_\varepsilon := \underline{(-\varepsilon, \varepsilon)}$.

Planar curvature as a stretch factor

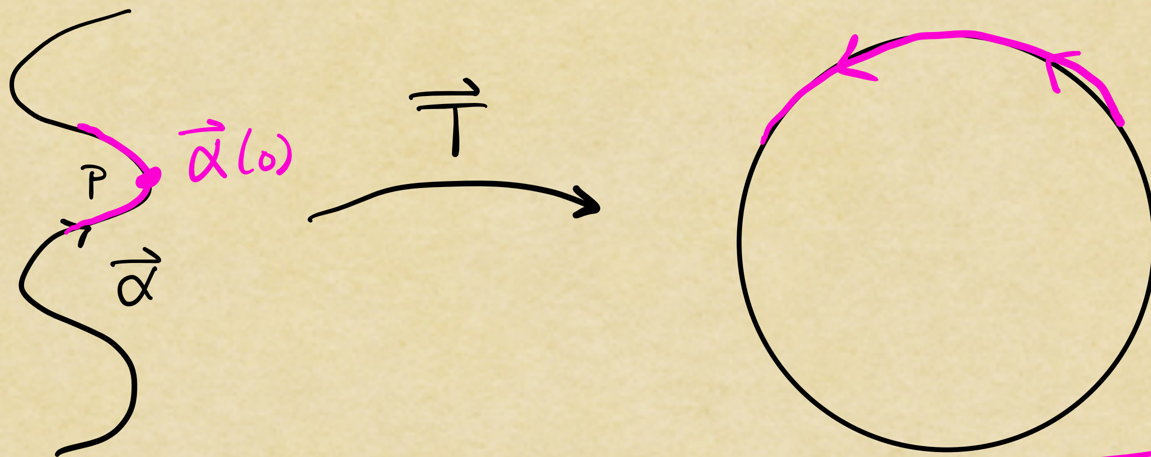


On $\mathrm{im}(\vec{\alpha})$ we have $\underline{\vec{\alpha}(I_\varepsilon)}$, which has length $\underline{\ell(\vec{\alpha}(I_\varepsilon))=2\varepsilon}$, since $\vec{\alpha}$ is unit-speed.

The length of $\underline{(\vec{T}\circ\vec{\alpha})(I_\varepsilon)} \subset S^1$ is given by

$$\int_{I_\varepsilon} \|(\vec{T}\circ\vec{\alpha})'(s)\|\, ds.$$

Planar curvature as a stretch factor



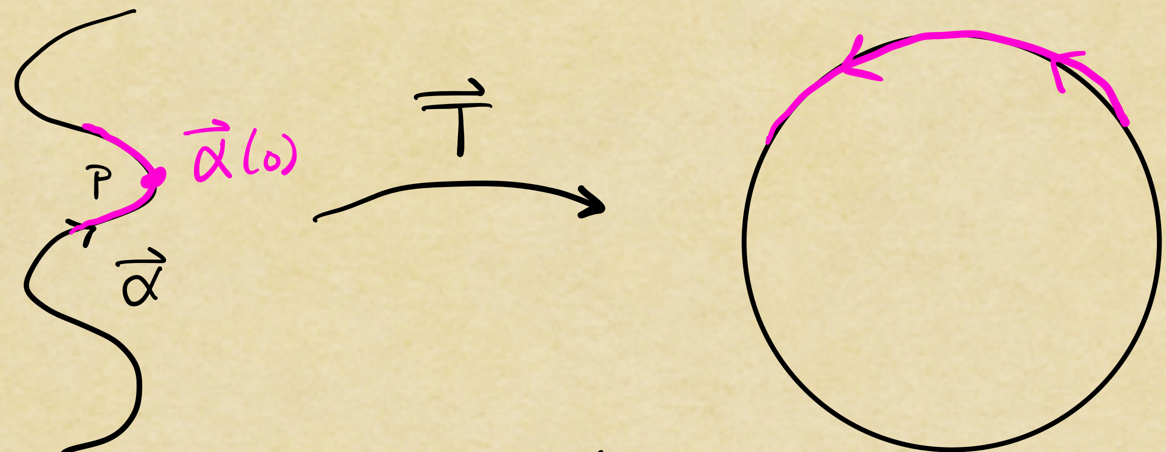
Abuse of notation

$$\ell\left((\vec{T}\circ\vec{\alpha})(I_\varepsilon)\right) = \int_{I_\varepsilon} \left\| (\vec{T}\circ\vec{\alpha})'(s) \right\| ds = \int_{-\varepsilon}^{\varepsilon} \left\| \vec{T}'(s) \right\| ds$$

$$= \int_{-\varepsilon}^{\varepsilon} |k(s)|\, ds \approx |k(0)| \cdot \ell(I_\varepsilon)$$

So $$\frac{\ell\left((\vec{T}\circ\vec{\alpha})(I_\varepsilon)\right)}{\ell\left(\vec{\alpha}(I_\varepsilon)\right)} \approx \frac{|k(0)| \cdot \ell(I_\varepsilon)}{\ell(I_\varepsilon)} = |k(0)|$$

Planar curvature as a stretch factor

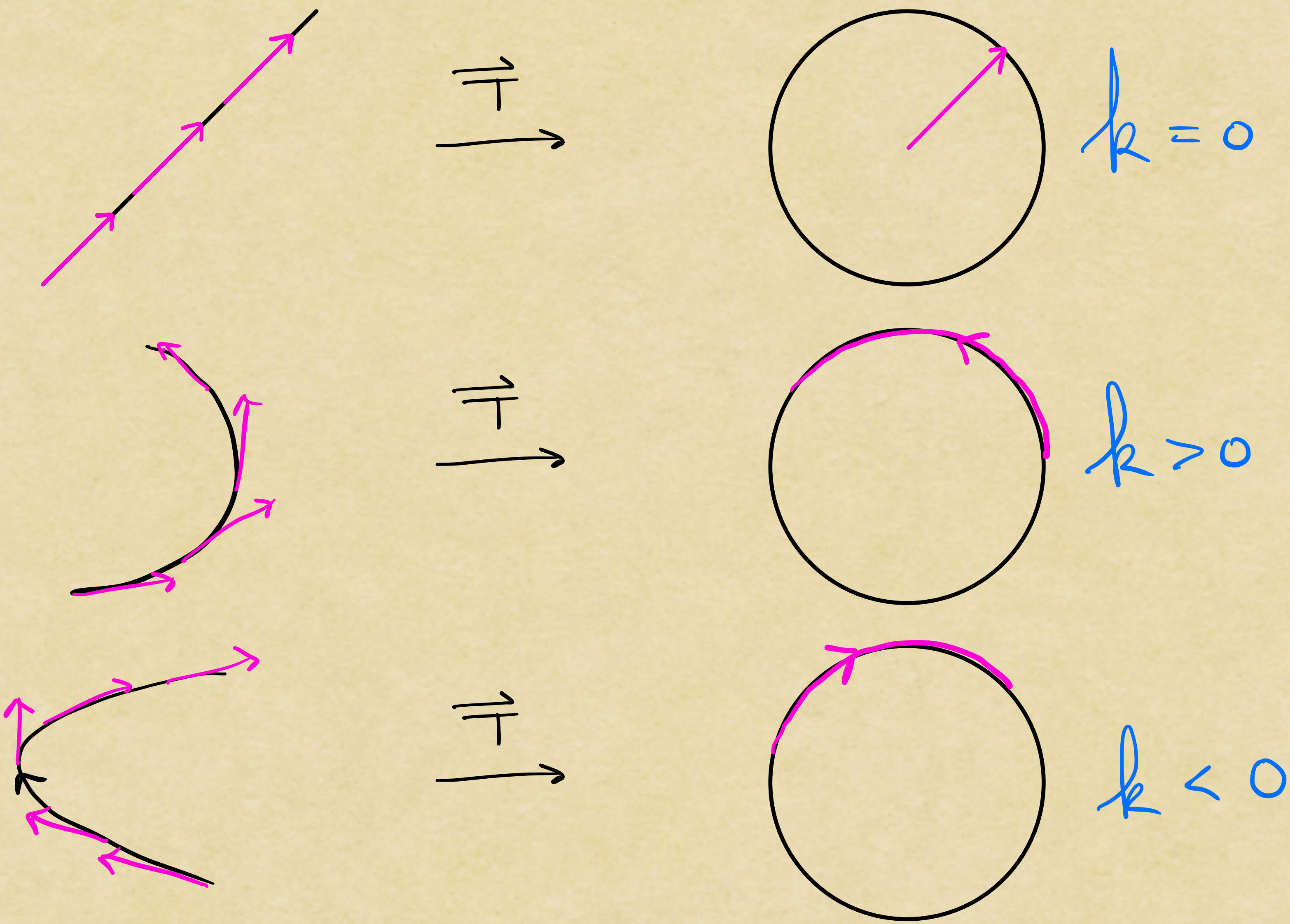


In particular, $\displaystyle \lim_{\varepsilon \to 0} \frac{\ell((\vec{T} \circ \vec{\alpha})(I_\varepsilon))}{\ell(\vec{\alpha}(I_\varepsilon))} = |k(0)|$,

so the planar curvature gives an instantaneous "stretch factor" for the tangent circular map.

The sign of k can be determined by orientations : CCW is positive.

Planar curvature as a stretch factor

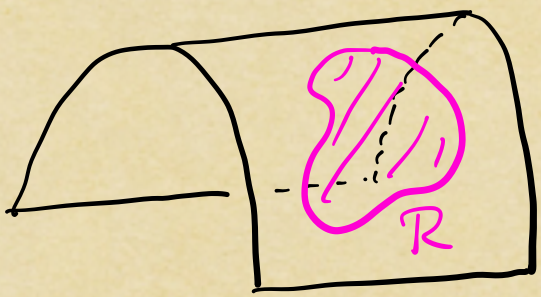


We'll interpret Gaussian curvature as a stretch factor for the Gauss map in much the same way.

This requires first thinking about surface integrals.

Surface integration

Given a function $f : \mathrm{im}(\vec{x}) \longrightarrow \mathbb{R}$, and a region $R \subset \mathrm{im}(\vec{x})$, we want $\iint_R f \, dA$.



$\xrightarrow{f} \mathbb{R}$

This is a <u>surface integral</u>, as seen in multivariable calculus.

$$\boxed{\iint_R f \, dA := \iint_{\vec{x}^{-1}(R)} (f \circ \vec{x}) \cdot \|\vec{x}_1 \times \vec{x}_2\| \, du^1 du^2}$$

Surface integration

Using a result from Activity 6, this is

$$\iint_R f\,dA := \iint_{\vec{x}^{-1}(R)} (f\circ\vec{x})\cdot\|\vec{x}_1\times\vec{x}_2\|\,du^1du^2$$

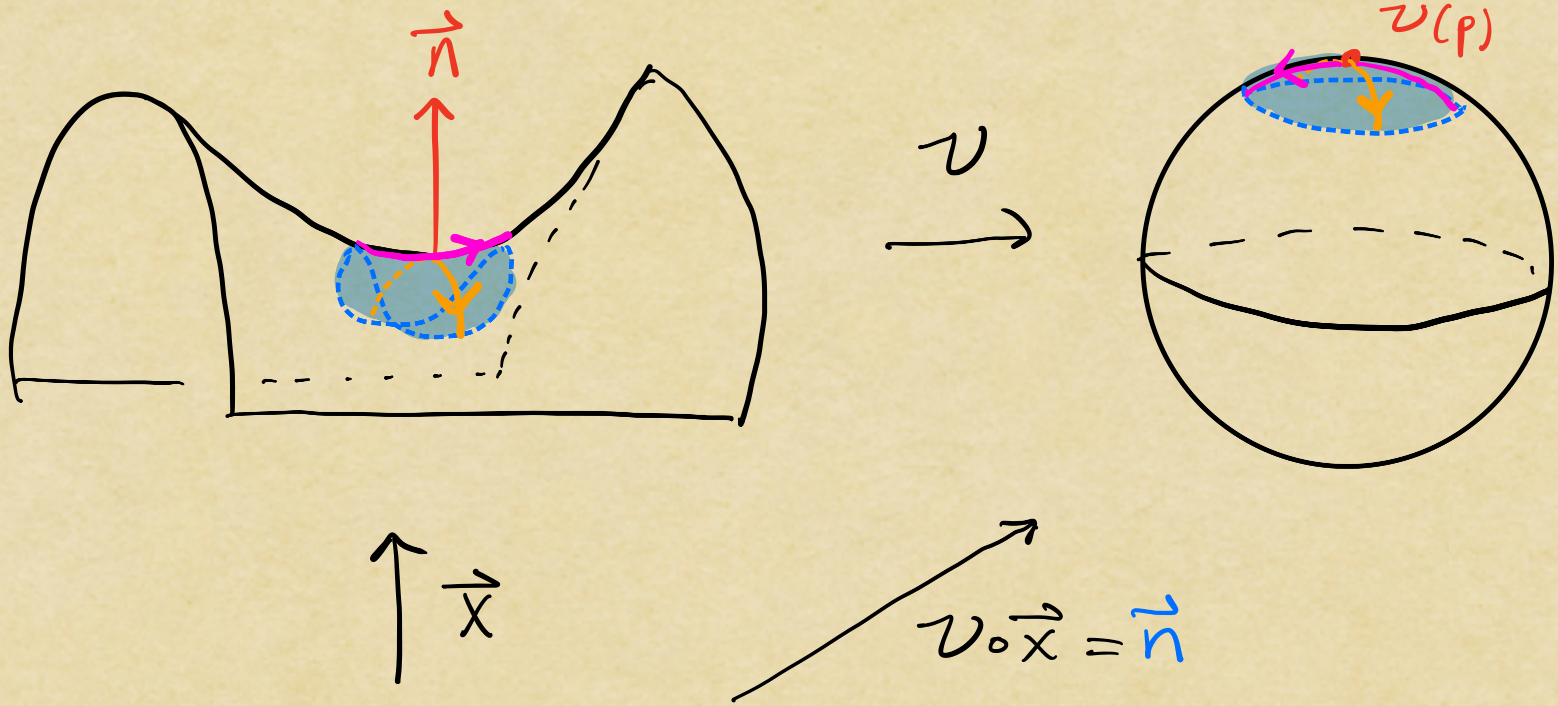
$$= \iint_{\vec{x}^{-1}(R)} (f\circ\vec{x})\sqrt{g}\,du^1du^2$$

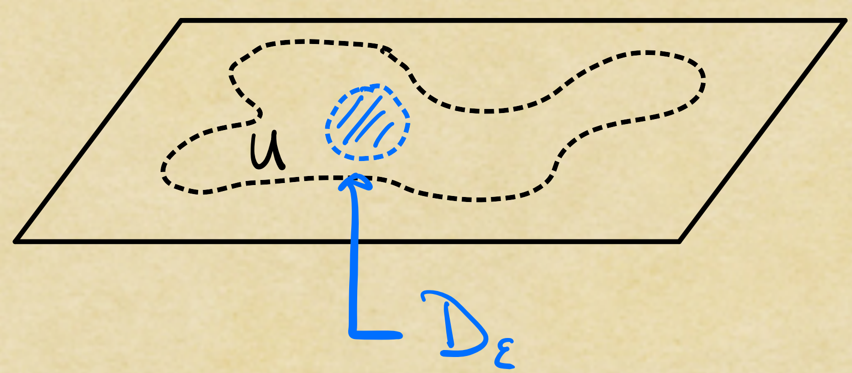
Example $\text{Area}(R) = \iint_R 1\,dA = \iint_{\vec{x}^{-1}(R)} \|\vec{x}_1\times\vec{x}_2\|\,du^1du^2$

$$= \iint_{\vec{x}^{-1}(R)} \sqrt{g}\,du^1du^2$$

The "fudge factor" for surface integrals is $\|\vec{x}_1\times\vec{x}_2\|$.

Gaussian curvature as a stretch factor





We want to compare $\underline{\text{Area}(\vec{n}(D_\varepsilon))}$ to $\underline{\text{Area}(\vec{x}(D_\varepsilon))}$.

Gaussian curvature as a stretch factor

$$\operatorname{Area}\left((\nu\circ\vec{x})(D_\varepsilon)\right) = \iint_{(\nu\circ\vec{x})^{-1}\left[(\nu\circ\vec{x})(D_\varepsilon)\right]} \left\| (\nu\circ\vec{x})_1 \times (\nu\circ\vec{x})_2 \right\| du^1\, du^2$$

$$= \iint_{\vec{n}^{-1}(\vec{n}(D_\varepsilon))} \left\| \vec{n}_1 \times \vec{n}_2 \right\| du^1\, du^2$$

Now $\vec{n}^{-1}(\vec{n}(D_\varepsilon)) = \underline{\quad D_\varepsilon \quad}$, and

$$\vec{n}_1 \times \vec{n}_2 = \left(-\mathcal{L}(\vec{x}_1)\right) \times \left(-\mathcal{L}(\vec{x}_2)\right)$$

$$= \left(-L_1^1 \vec{x}_1 - L_1^2 \vec{x}_2\right) \times \left(-L_2^1 \vec{x}_1 - L_2^2 \vec{x}_2\right)$$

$$= \left(L_1^1 L_2^2 - L_1^2 L_2^1\right) \vec{x}_1 \times \vec{x}_2 \qquad (\vec{x}_i \times \vec{x}_i = \vec{0})$$

$$= \det\left(L_j^i\right) \vec{x}_1 \times \vec{x}_2 = K\ \vec{x}_1 \times \vec{x}_2$$

Gaussian curvature as a stretch factor

So $\text{Area}\left((\nu\circ\vec{x})(D_\varepsilon)\right) = \iint_{D_\varepsilon} \|\vec{n}_1 \times \vec{n}_2\|\, du^1 du^2$

$$= \iint_{D_\varepsilon} \|K(\vec{x}_1 \times \vec{x}_2)\|\, du^1 du^2$$

$$\approx |K(0)| \iint_{D_\varepsilon} \|\vec{x}_1 \times \vec{x}_2\|\, du^1 du^2$$

$$= |K(0)| \cdot \text{Area}\left(\vec{x}(D_\varepsilon)\right)$$

Gaussian curvature as a stretch factor

Upshot: $\dfrac{\text{Area}((\nu \circ \vec{x})(D_\varepsilon))}{\text{Area}(\vec{x}(D_\varepsilon))} \approx |K(p)|$

Concretely, $\boxed{\lim_{\varepsilon \to 0} \dfrac{\text{Area}((\nu \circ \vec{x})(D_\varepsilon))}{\text{Area}(\vec{x}(D_\varepsilon))} = |K(p)|}$

So Gaussian curvature tells us the factor by which the Gauss map distorts areas.

The sign is determined by whether or not ν preserves orientation.

Gaussian curvature as a stretch factor