Math 4441 **November 2, 2022**

LAST TIME

Length-minimizing curves in the plane are geodesics.

TODAY

Prove the same result on a simple surface.

$\boxed{\begin{array}{l}\underline{\text{Thm}}\ \text{If } \vec{\alpha} = \vec{X} \circ \vec{\alpha}_u \text{ is a unit-speed surface curve which minimizes arc} \\ \text{length between points } \vec{\alpha}(a) \text{ \& } \vec{\alpha}(b), \text{ then } \kappa_g(s) = 0, \text{ for all } s \in (a,b).\end{array}}$

We'll break the proof into two steps:

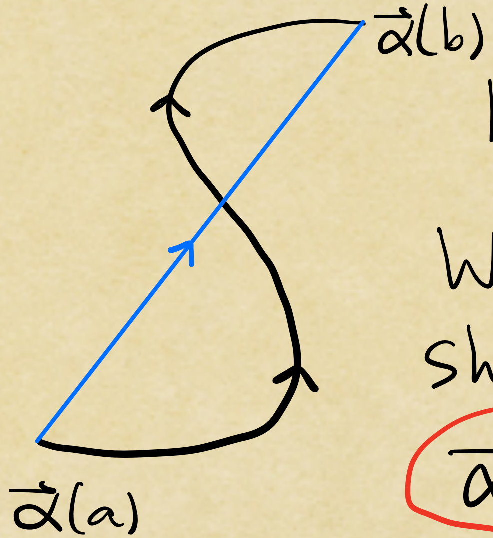
1. prove the <u>planar</u> version (using <u>Calculus of variations</u>);
2. figure out how to recreate the argument on a surface.

Where's the pinch point when we try to graduate from the planar case to a surface?

Recap.

The key idea was that if $\kappa_g(s) \neq 0$ for some $s \in (a,b)$, then we could find a shorter path by <u>pushing towards $\kappa_g J\vec{T}$</u>.

$\vec{S}$



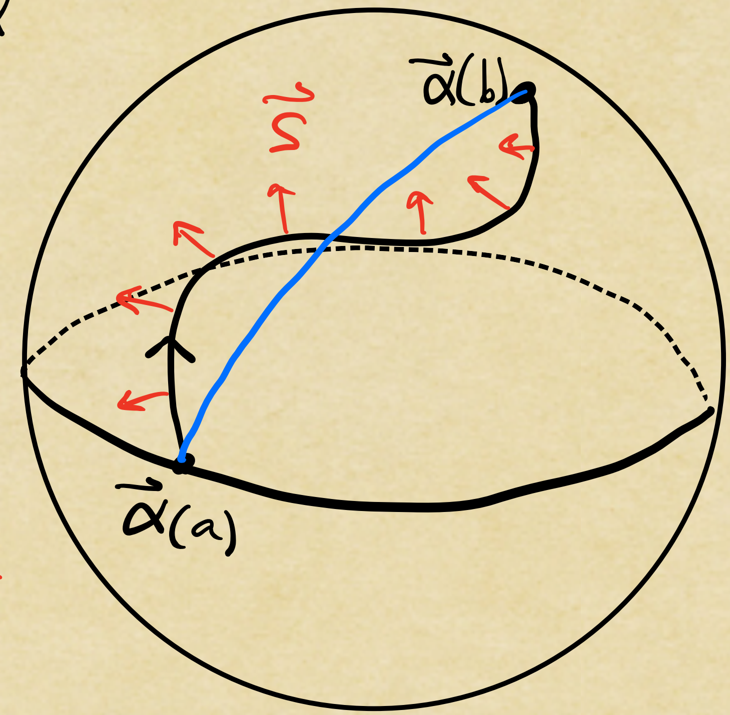
We assumed $\vec{\alpha}$ to be the shortest, built the family

$$\vec{\alpha}_t := \vec{\alpha} + t\,\kappa_g J\vec{T},$$

Not in surface!

and used IBP to find that $\kappa_g \equiv 0$.

Instead of pushing towards $k_g J\vec{T}$, we make $\vec{\alpha}$ shorter by "pushing towards" $k_g \vec{S}$.



But we can't just define

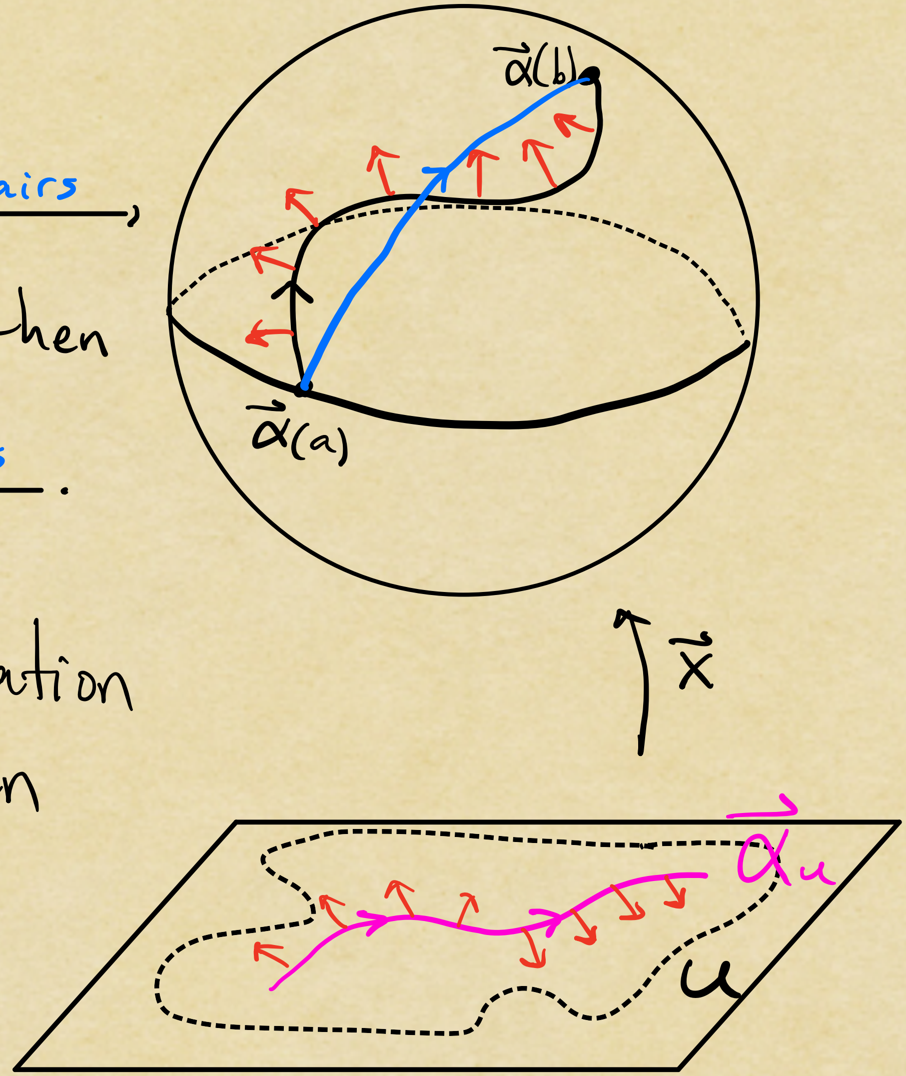
$$\vec{\alpha}_t := \cancel{\vec{\alpha} + t \cdot k_g \vec{S}}$$

This won't be in the surface!

Solution

Take things downstairs, then "just push," then come back upstairs.

But! Our computation still has to happen upstairs.



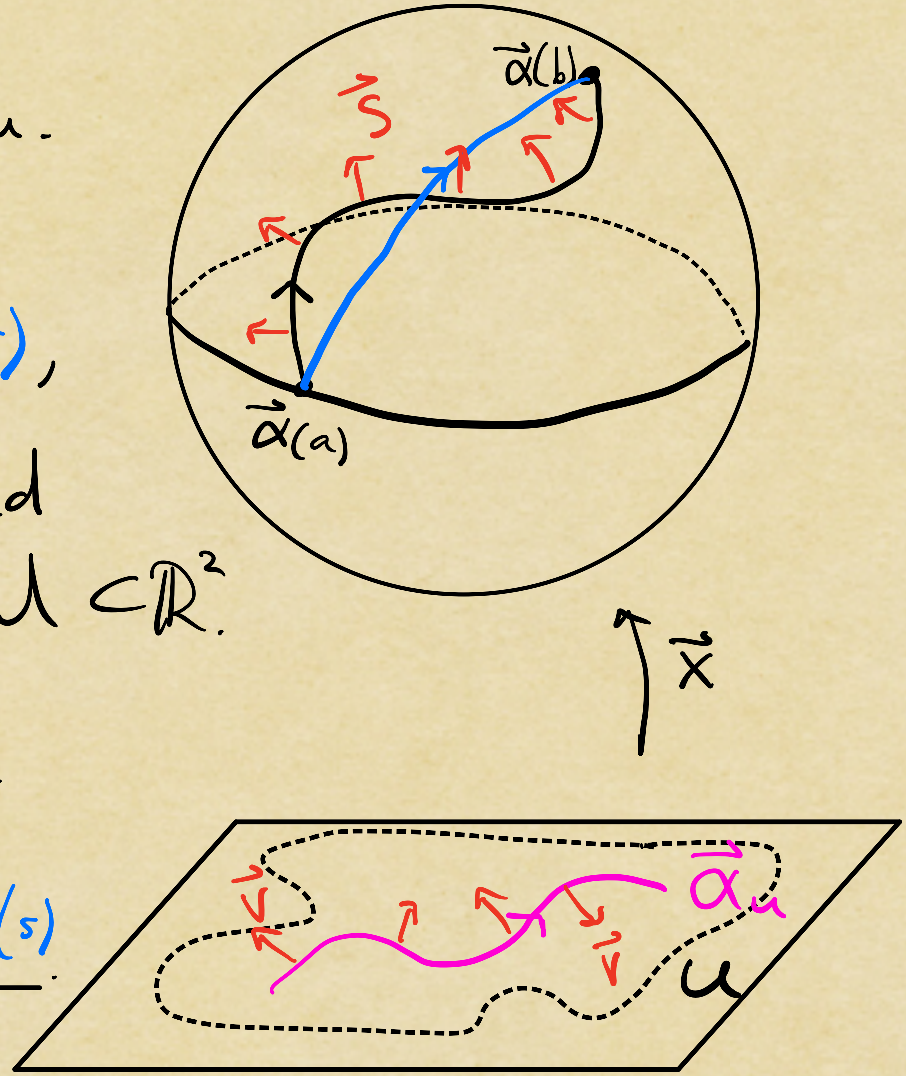
Solution

We have $\vec{\alpha} = \vec{x} \circ \vec{\alpha}_u$.

We want

$$\vec{S}(s) = d\vec{x}_{\vec{\alpha}_u(s)} \vec{V}(s),$$

for some vector field $\vec{V}(s)$ along $\vec{\alpha}_u \subset U \subset \mathbb{R}^2$.



No problem: set

$$\vec{V}(s) = \left(d\vec{x}_{\vec{\alpha}_u(s)}\right)^{-1} \vec{S}(s)$$

Solution

We have $\vec{\alpha} = \vec{x} \circ \vec{\alpha}_u$

and $\vec{V} = d\vec{x}^{-1} \circ \vec{S}$.

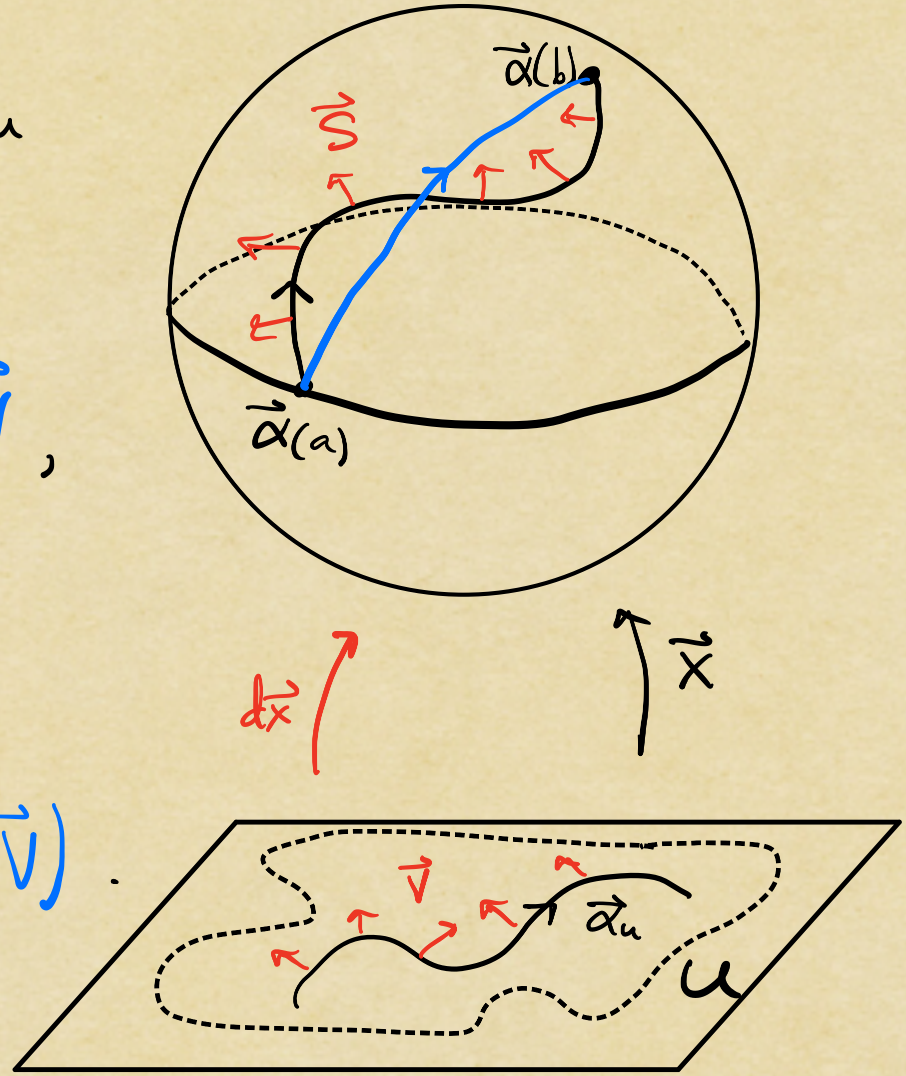
Set

$$\vec{\alpha}_{u,t} := \vec{\alpha}_u + t \cdot k_g \vec{V},$$

and then set

$$\vec{\alpha}_t := \vec{x} \circ \vec{\alpha}_{u,t}$$

$$= \vec{x}(\vec{\alpha}_u + t\, k_g \vec{V}).$$



Once we have

$$\vec{\alpha}_t = \vec{x}\left(\vec{\alpha}_u + t\, k_g \vec{V}\right),$$

the strategy works as in the planar case.

Set $L(t) = \ell(\vec{\alpha}_t) = \int_a^b \|\vec{\alpha}_t{}'(s)\|\, ds$

Since $\vec{\alpha}_0 = \underline{\ \vec{\alpha}\ }$ and $\vec{\alpha}$ is length-minimizing,

$$0 = \underline{L'(0)} = \frac{\partial}{\partial t}\left(\int_a^b \sqrt{\left\langle \frac{\partial}{\partial s}\vec{\alpha}_t, \frac{\partial}{\partial s}\vec{\alpha}_t \right\rangle}\, ds\right)\Bigg|_{t=0}$$

$$= \int_a^b \frac{\partial}{\partial t}\left(\sqrt{\left\langle \frac{\partial}{\partial s}\vec{\alpha}_t, \frac{\partial}{\partial s}\vec{\alpha}_t \right\rangle}\right)\Bigg|_{t=0} ds$$

$$= \int_a^b \frac{\frac{\partial}{\partial t}\left\langle \frac{\partial}{\partial s}\vec{\alpha}_t, \frac{\partial}{\partial s}\vec{\alpha}_t \right\rangle}{2\sqrt{\left\langle \frac{\partial}{\partial s}\vec{\alpha}_t, \frac{\partial}{\partial s}\vec{\alpha}_t \right\rangle}}\Bigg|_{t=0} ds$$

$$= \int_a^b \frac{\cancel{2}\left\langle \frac{\partial^2}{\partial t \partial s}\vec{\alpha}_t, \frac{\partial}{\partial s}\vec{\alpha}_t \right\rangle + \cancel{\left\langle \frac{\partial}{\partial s}\vec{\alpha}_t, \frac{\partial^2}{\partial t \partial s}\vec{\alpha}_t \right\rangle}}{\cancel{2}\sqrt{\left\langle \frac{\partial}{\partial s}\vec{\alpha}_t, \frac{\partial}{\partial s}\vec{\alpha}_t \right\rangle}}\Bigg|_{t=0} ds$$

1, b/c $\vec{\alpha}_0 = \vec{\alpha}$ is unit-speed

$$= \int_a^b \left\langle \frac{\partial^2}{\partial t \partial s}\vec{\alpha}_t, \frac{\partial}{\partial s}\vec{\alpha}_t \right\rangle\Bigg|_{t=0} ds$$

IBP

$$= \int_a^b \left[\frac{\partial}{\partial s}\left(\overbrace{\left\langle \underbrace{\frac{\partial}{\partial t}\vec{\alpha}_t}_{\kappa_g \vec{S}}, \underbrace{\frac{\partial}{\partial s}\vec{\alpha}_t}_{\vec{\alpha}_t'} \right\rangle}^{0} \right) - \left\langle \frac{\partial}{\partial t}\vec{\alpha}_t, \frac{\partial^2}{\partial s^2}\vec{\alpha}_t \right\rangle \right]_{t=0} ds$$

From here, the calculus steps match last time.

$$0 = L'(0) = \int_a^b - \left\langle \frac{\partial}{\partial t}\vec{\alpha}_t, \frac{\partial^2}{\partial s^2}\vec{\alpha}_t \right\rangle \Big|_{t=0} ds$$

Since $\vec{\alpha}_t = \vec{x}(\vec{\alpha}_u + t\kappa_g \vec{V})$,

$$\frac{\partial}{\partial t}\vec{\alpha}_t = d\vec{x}_{\vec{\alpha}_u + t\kappa_g\vec{V}}(\kappa_g \vec{V}) = \kappa_g \cdot d\vec{x}\,\vec{V} = \kappa_g \vec{S}$$

Since $\vec{\alpha}_0 = \vec{\alpha}$, $\quad \frac{\partial^2}{\partial s^2}\vec{\alpha}_t \Big|_{t=0} = \vec{\alpha}''$.

So $0 = L'(0) = \int_a^b - \langle \kappa_g \vec{S}, \vec{\alpha}''\rangle\, ds$

$$= \int_a^b -\kappa_g \langle \vec{S}, \vec{\alpha}''\rangle\, ds = \int_a^b -\kappa_g \cdot \kappa_g\, ds$$

Altogether, $0 = L'(0) = \int_a^b -\kappa_g^2 \, ds.$

So we must have $\kappa_g(s) = 0$, for all $s \in (a,b)$.

So $\vec{\alpha}$ is a geodesic !

Morally, we did the planar argument again. We just had to be careful about "pushing off."