Math 4441 November 16, 2022

LAST TIME

We repackaged the Weingarten map $\mathcal{L} : \underline{T_pS} \to \underline{T_pS}$ as the second fundamental form

$$\mathrm{II}_p : \underline{T_pS \times T_pS} \longrightarrow \underline{\mathbb{R}}$$

and presented the formula $\underline{(L_{ij}) = (g_{ij})(L^i_j)}$.

TODAY

We'll extract from this data a single number which we'll call the <u>curvature</u> of our surface — namely, the <u>Gaussian curvature</u>.

Recollections from Monday

$$\mathrm{I\!I}(\vec{X},\vec{Y}) := \langle \vec{X}, \mathcal{L}(\vec{Y}) \rangle$$

$$[\mathrm{I\!I}]_{\{\vec{x}_1,\vec{x}_2\}} = (L_{ij}) \quad ; \quad [\mathcal{L}]_{\{\vec{x}_1,\vec{x}_2\}} = (L^i_j)$$

These record the same data, but sometimes we prefer (L_{ij}) because it's symmetric.

What numerical information might we want from these matrices?

determinant, trace $\rightsquigarrow$ eigenvalues

What would the eigenvalues of $\mathcal{L}$ tell us?

Giving a geometric answer to this question requires backing up a bit.

Recall our observation on Monday: for $\vec{\alpha}$ a surface curve* $\langle \vec{T}, \vec{n} \rangle \equiv 0$ implies

* not nec. unit speed

$$0 = D_{\vec{T}} \langle \vec{T}, \vec{n} \rangle = \langle D_{\vec{T}} \vec{T}, \vec{n} \rangle + \langle \vec{T}, D_{\vec{T}} \vec{n} \rangle$$

$$0 = \left\langle \frac{d}{ds} \vec{T}, \vec{n} \right\rangle + \langle \vec{T}, -\mathcal{L}(\vec{T}) \rangle$$

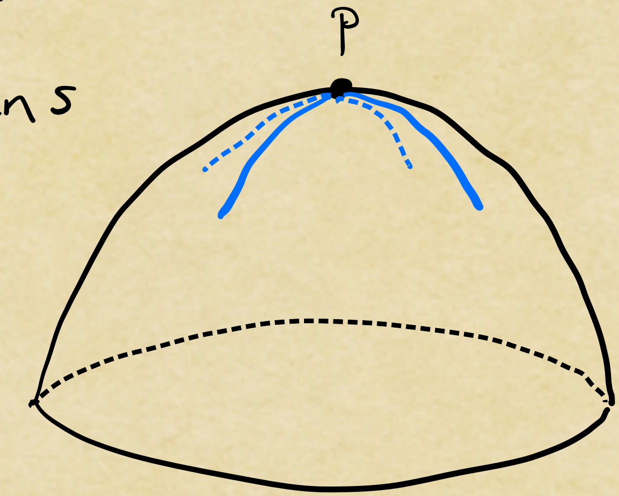
$$\therefore \quad \langle \vec{T}, \mathcal{L}(\vec{T}) \rangle = \left\langle \frac{d}{ds} \vec{T}, \vec{n} \right\rangle$$

So $\mathrm{II}(\vec{T}, \vec{T}) = \left\langle \frac{d}{ds} \vec{T}, \vec{n} \right\rangle = \kappa_n$

plugging a unit vector into II gives a κ_n

So $\mathrm{II}(\vec{T},\vec{T}) = \langle \frac{d}{ds}\vec{T}, \vec{n} \rangle = \kappa_n.$

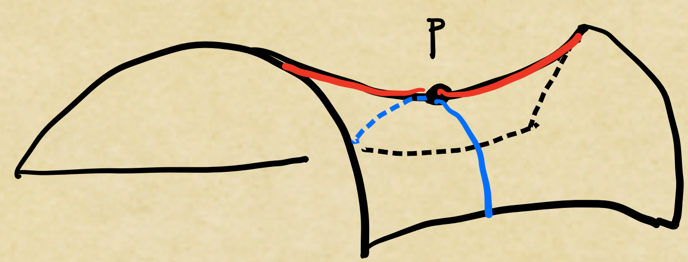
This is nice because it means



① the normal curvature of $\vec{\alpha}$ depends only on $\underline{\ \vec{T}\ }$;

② We can study $\underline{\text{all}}$ $\underline{\text{normal curvatures}}$ out of p by studying the map $T_pS \to \mathbb{R}$

$$\frac{\vec{v} \mapsto \mathrm{II}(\vec{v},\vec{v})}{(\|\vec{v}\| = 1)}$$



So we have a new way of studying the normal curvatures: study the map

$$\vec{v} \mapsto \mathrm{II}(\vec{v}, \vec{v}).$$

Now suppose we could find an ONB $\{\vec{v}_1, \vec{v}_2\}$ for $T_p \operatorname{im} \vec{x}$ such that

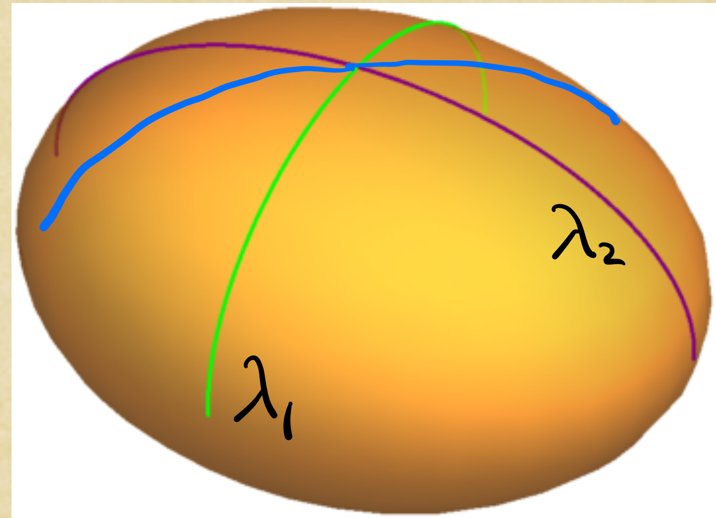
$$\mathcal{L}(\vec{v}_1) = \lambda_1 \vec{v}_1 \quad ; \quad \mathcal{L}(\vec{v}_2) = \lambda_2 \vec{v}_2.$$

Then

$$\mathrm{II}(\vec{v}_i, \vec{v}_i) = \langle \vec{v}_i, \mathcal{L}(\vec{v}_i) \rangle = \langle \vec{v}_i, \lambda_i \vec{v}_i \rangle$$
$$= \lambda_i \langle \vec{v}_i, \vec{v}_i \rangle = \lambda_i$$

So the <u>eigenvalues</u> of $\mathcal{L}$ would be <u>normal curvatures</u>!

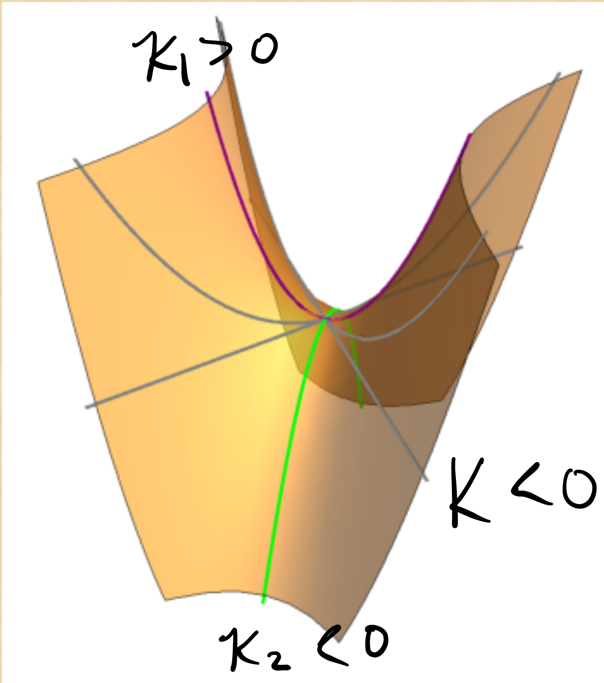
Now every normal curvature out of p can be written as $\underline{\mathrm{II}(\vec{v},\vec{v})}$ for some $\vec{v}$ with $\|\vec{v}\| = 1$.



Then $\vec{v} = \underline{\cos\theta\, \vec{v}_1 + \sin\theta\, \vec{v}_2}$.

So
$$\mathrm{II}(\vec{v},\vec{v}) = \langle \cos\theta\, \vec{v}_1 + \sin\theta\, \vec{v}_2, \mathcal{L}(\cos\theta\, \vec{v}_1 + \sin\theta\, \vec{v}_2)\rangle$$
$$= \langle \cos\theta\, \vec{v}_1 + \sin\theta\, \vec{v}_2, \lambda_1 \cos\theta\, \vec{v}_1 + \lambda_2 \sin\theta\, \vec{v}_2\rangle$$
$$= \lambda_1 \cos^2\theta + \lambda_2 \sin^2\theta \quad (\text{b/c ONB})$$

This means that $\underline{\lambda_2} \leq \kappa_n \leq \underline{\lambda_1}$. The eigenvalues of $\mathcal{L}$ are <u>extreme values for κ_n</u>!



So if $\mathcal{L}$ admits an orthonormal eigenbasis $\{\vec{v}_1, \vec{v}_2\}$, we can write

$$[\mathcal{L}]_{\{\vec{v}_1, \vec{v}_2\}} = \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix},$$

with $\kappa_1 \geqslant \kappa_2$.

We call $\vec{v}_1$ & $\vec{v}_2$ the principal directions at p, and κ_1 & κ_2 the principal curvatures.

The Gaussian curvature at p is given by

$$K := \det \mathcal{L}_p = \kappa_1 \kappa_2 .$$

But wait! Why should $\mathcal{L}$ admit an orthonormal eigenbasis? This follows from the <u>spectral theorem</u>.

Thm. A linear transformation $T: V \to V$ on an inner product space admits an orthonormal eigenbasis iff T is self-adjoint:

$$\langle T(\vec{v}), \vec{w} \rangle = \langle \vec{v}, T(\vec{w}) \rangle,$$

for all $\vec{v}, \vec{w}$.

We can check that $\mathcal{L}$ is self-adjoint:

$$\langle \mathcal{L}(\vec{v}), \vec{w} \rangle = \langle \vec{w}, \mathcal{L}(\vec{v}) \rangle = \mathrm{II}(\vec{w}, \vec{v}) = \mathrm{II}(\vec{v}, \vec{w})$$

b/c (L_{ij}) is symm.

$$= \langle \vec{v}, \mathcal{L}(\vec{w}) \rangle.$$

So what have we done?

① If $\vec{v} \in T_pS$ is a unit vector, then $\mathrm{II}_p(\vec{v},\vec{v})$ is a normal curvature.

② Because II is symmetric, $\mathcal{L}$ is self-adjoint, and thus admits an orthonormal eigenbasis $\{\vec{v}_1, \vec{v}_2\}$.

③ The eigenvalues $\kappa_1 \geq \kappa_2$ of $\mathcal{L}$ give extreme values for κ_n at p.

④ The Gaussian curvature at p is defined to be $K(p) := \det \mathcal{L}_p = \kappa_1 \cdot \kappa_2$.

Example We've previously seen that any simple surface with image on a sphere of radius R will have

$$(L^i_j) = \pm \begin{pmatrix} 1/R & 0 \\ 0 & 1/R \end{pmatrix}.$$

The principal curvatures are thus

$$\kappa_1 = 1/R \quad , \quad \kappa_2 = 1/R \quad ,$$

and the Gaussian curvature is

$$K = \frac{1}{R^2} .$$

Note that <u>every direction</u> is principal.

Computing K from fundamental forms

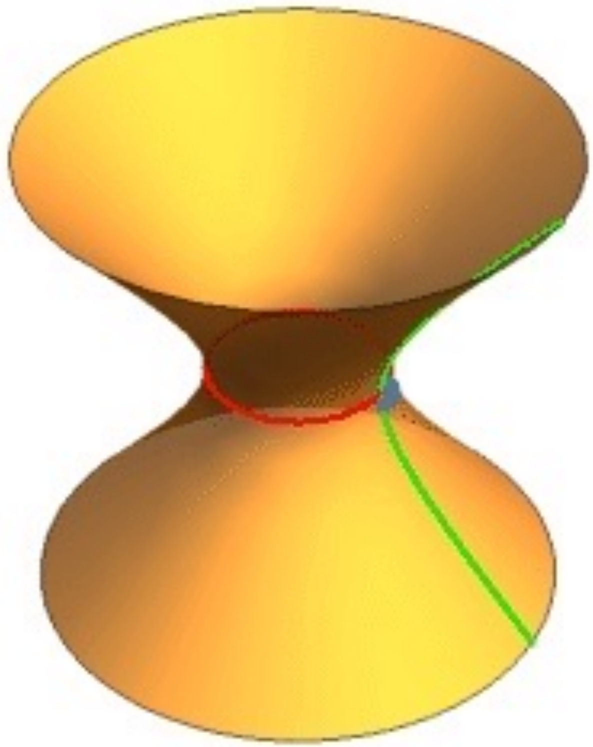
Recall that $(L_{ij}) = \underline{(g_{ij})(L^i_j)}$.

So $\det(L_{ij}) = \det(g_{ij}) \cdot K$

$$\Rightarrow K = \frac{\det(L_{ij})}{\det(g_{ij})}.$$

This is sometimes written as

$$\boxed{K = \frac{\det(\mathrm{II})}{\det(\mathrm{I})}.}$$



$K(p) = -1.$