**Math 4441** **November 14, 2022**

## LAST TIME

We defined the Weingarten map (AKA shape operator) $\mathcal{L}_p : T_pS \to T_pS$, with the goal of measuring the curvature of the surface. Also,

I left this off in class!

$$\vec{X}_{ij} = L_{ij}\vec{n} + \sum_{k=1}^{2} \Gamma_{ij}^{k} \vec{X}_k \quad ; \quad \vec{n}_j = -\sum_{i=1}^{2} L_j^i \vec{X}_i .$$

Thanks, Juana!

## TODAY

We'll see that $(L_{ij})$ and $(L_j^i)$ encode the same information. Specifically, we'll repackage $\mathcal{L}_p$ as $\mathrm{II}_p : T_pS \times T_pS \to \mathbb{R}$.

Let's recall the definitions:

$L_{ij} := \langle \vec{x}_{ij}, \vec{n} \rangle$

$\rightsquigarrow (L_{ij}) =$ "how much does $\vec{x}$ bend towards $\vec{n}$?"

$\mathcal{L}_p(\vec{v}) := -D_{\vec{v}} \nu$

= "how does $\vec{n}$ $(= \nu)$ change as we move towards $\vec{v}$ ?"

The real link between these is <u>linear algebra</u>, but let's motivate it with <u>normal curvature</u>.

(And also keep a promise about minus signs.)

# Why the minus sign? (take #2)

When studying surface curves, we said that $\underline{K_n}$ tells us more about the surface than it does about the curve.

Let's investigate.

For any surface curve $\vec{\alpha}$, $\langle \vec{\alpha}', \vec{n} \rangle \equiv \underline{0}$.

We can differentiate in the direction $\vec{\alpha}'$:

$$D_{\vec{\alpha}'} \langle \vec{\alpha}', \vec{n} \rangle = \langle D_{\vec{\alpha}'} \vec{\alpha}', \vec{n} \rangle + \langle \vec{\alpha}', D_{\vec{\alpha}'} \vec{n} \rangle$$

sloppy

$$D_{\vec{\alpha}'}\langle \vec{\alpha}', \vec{n}\rangle = \langle D_{\vec{\alpha}'}\vec{\alpha}', \vec{n}\rangle + \langle \vec{\alpha}', D_{\vec{\alpha}'}\vec{n}\rangle$$

So $\quad 0 = \langle \vec{\alpha}'', \vec{n}\rangle + \langle \vec{\alpha}', -\mathcal{L}(\vec{\alpha}')\rangle$

Then $\langle \vec{\alpha}', \mathcal{L}(\vec{\alpha}')\rangle = \langle \vec{\alpha}'', \vec{n}\rangle$

$$= k_n$$

This might not convince you that we baked the minus sign into the right thing, but hopefully you see that a minus sign must appear *somewhere.*

Notice that extracting $K_n$ required thinking about $\underline{\langle \vec{\alpha}', \mathcal{L}(\vec{\alpha}') \rangle}$. This is where the linear algebra snuck in.

If $V$ is an <u>inner product space</u> and

$$T : V \to V$$

is a linear transformation, we can always define a <u>bilinear form</u>

$$B_T : V \times V \longrightarrow \mathbb{R}$$

by $B_T(\vec{v}, \vec{w}) := \langle \vec{v}, T(\vec{w}) \rangle$.

Sometimes we prefer working with $B_T$ to $T$.

**Def** The second fundamental form of a surface $S$ at a point $p \in S$ is the bilinear form associated to $\mathcal{L}_p$. That is,

$$\mathrm{II}_p : T_pS \times T_pS \longrightarrow \mathbb{R}$$

is defined by

$$\mathrm{II}_p(\vec{X}, \vec{Y}) := \langle \vec{X}, \mathcal{L}(\vec{Y}) \rangle ,$$

for all $\vec{X}, \vec{Y} \in T_pS$.

## Remarks

1. Unlike $\mathrm{I}_p$, $\mathrm{II}_p$ is not an inner product.

2. The matrix $[\mathrm{II}_p]_{\{\vec{X}_1, \vec{X}_2\}}$ is nicer than $[\mathcal{L}_p]_{\{\vec{X}_1, \vec{X}_2\}}$.

# The matrix representation for $\mathrm{II}$

We want to represent the form

$$\mathrm{II}_p : T_pS \times T_pS \longrightarrow \mathbb{R}$$

with a matrix $[\mathrm{II}_p]_{\{\vec{x}_1, \vec{x}_2\}}$.

How should we build this matrix?

Hint: How did we build the matrix representation of $I_p$?

$ij$-entry : $\mathrm{II}_p(\vec{x}_i, \vec{x}_j)$

## The matrix representation for $\mathrm{II}$

The matrix $[\mathrm{I}_p]_{\{\vec{x}_1,\vec{x}_2\}}$ is $(g_{ij})$, and we have

$$\mathrm{I}_p(\vec{X},\vec{Y}) = \begin{pmatrix} X^1 \\ X^2 \end{pmatrix}^T \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \begin{pmatrix} Y^1 \\ Y^2 \end{pmatrix}.$$

Likewise, we want

$$\mathrm{II}_p(\vec{X},\vec{Y}) = \begin{pmatrix} X^1 \\ X^2 \end{pmatrix}^T [\mathrm{II}_p]_{\{\vec{x}_1,\vec{x}_2\}} \begin{pmatrix} Y^1 \\ Y^2 \end{pmatrix}.$$

So the ij-entry of $[\mathrm{II}_p]_{\{\vec{x}_1,\vec{x}_2\}}$ will be $\underline{\mathrm{II}_p(\vec{x}_i,\vec{x}_j)}$.

## The matrix representation for $\mathrm{II}$

The ij-entry of $[\mathrm{II}_p]_{\{\vec{x}_1,\vec{x}_2\}}$ will be

$$\mathrm{II}_p(\vec{x}_i,\vec{x}_j) = \langle \vec{x}_i, \mathcal{L}(\vec{x}_j)\rangle$$

At the same time, $\langle \vec{x}_i, \vec{n}\rangle \equiv 0$.

So $0 = \langle \vec{x}_{ij}, \vec{n}\rangle + \langle \vec{x}_i, \vec{n}_j\rangle$

$$= L_{ij} + \langle \vec{x}_i, -\mathcal{L}(\vec{x}_j)\rangle \quad .$$

$\therefore \quad \langle \vec{x}_i, \mathcal{L}(\vec{x}_j)\rangle = L_{ij}$

i.e., $\mathrm{II}_p(\vec{x}_i,\vec{x}_j) = L_{ij}$ .

# The matrix representation for $\mathrm{II}$

**Upshot:** $[\mathrm{II}_p]_{\{\vec{x}_1, \vec{x}_2\}} = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}$

**Remarks**

1. This is why we called $L_{ij}$ the coefficients of the second fundamental form.
2. The matrix $(L_{ij})$ is better than $(L^i_j)$ because it is symmetric.
3. The matrices $(L_{ij})$, $(g_{ij})$, & $(L^i_j)$ are related by $\boxed{(L_{ij}) = (g_{ij})(L^i_j)}$.

## Proof of formula (time permitting) ← On HW 6 instead. Leaving outline here as a guide.

Given $\vec{X} = X^1\vec{x}_1 + X^2\vec{x}_2$ & $\vec{Y} = Y^1\vec{x}_1 + Y^2\vec{x}_2$,

the matrices satisfy

$$\begin{pmatrix} X^1 \\ X^2 \end{pmatrix}^T (g_{ij}) \begin{pmatrix} Y^1 \\ Y^2 \end{pmatrix} = \qquad\qquad\qquad ,$$

$$(L^i_j)\begin{pmatrix} Y^1 \\ Y^2 \end{pmatrix} = \qquad\qquad\qquad ,$$

& $$\begin{pmatrix} X^1 \\ X^2 \end{pmatrix}^T (L_{ij}) \begin{pmatrix} Y^1 \\ Y^2 \end{pmatrix} =$$

$$= \qquad\qquad\qquad .$$

## Proof of formula (time permitting)

So

$$\begin{pmatrix} X^1 \\ X^2 \end{pmatrix}^T (L_{ij}) \begin{pmatrix} Y^1 \\ Y^2 \end{pmatrix} = \langle \vec{X}, \mathcal{L}(\vec{Y}) \rangle$$

$$=$$

$$=$$

$\diamond$