Math 4441 Midterm 2

November 21, 2022

Name: ______________________ **gtID**: ______________________

Instructions. Read each question carefully and show all your work. Answers without justification will receive little to no credit. Writing your answers in a legible, well-organized manner will maximize your opportunities for partial credit.

This is a closed-note, closed-book exam, and you are expected to abide by the Georgia Tech Honor Challenge. Good luck!

Clearly label any extra papers you want graded.

By signing below, I certify that all work submitted on this exam is my own, and that I have neither given nor received any unauthorized help on this exam.

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Question	Points	Score
1	10	
2	10	
3	10	
4	10	
Total:	40	

Formulas

- $df_p = \begin{pmatrix} \frac{\partial f^1}{\partial u^1}(p) & \cdots & \frac{\partial f^1}{\partial u^m}(p) \\ \vdots & \ddots & \vdots \\ \frac{\partial f^n}{\partial u^1}(p) & \cdots & \frac{\partial f^n}{\partial u^m}(p) \end{pmatrix}$
- $d(g \circ f)_p = dg_{f(p)} \cdot df_p$
- $\frac{\partial (g\circ f)^i}{\partial u^j}|_p = \sum_{k=1}^m \frac{\partial g^i}{\partial v^k}|_{f(p)} \frac{\partial f^k}{\partial u^j}|_p$
- $\vec{x}_i := \frac{\partial \vec{x}}{\partial u^i}$
- $\tilde{\vec{x}} = \vec{x} \circ F$ implies $\tilde{\vec{x}}_j = \sum_{i=1}^2 \frac{\partial F^i}{\partial \tilde{u}^j} \vec{x}_j$
- $\vec{n} := \frac{\vec{x}_1 \times \vec{x}_2}{\|\vec{x}_1 \times \vec{x}_2\|}$
- $I_p(\vec{X}, \vec{Y}) := \langle \vec{X}, \vec{Y} \rangle_{\mathbb{R}^3}$
- $g_{ij} = \langle \vec{x}_i, \vec{x}_j \rangle$
- $I_p(\vec{X}, \vec{Y}) = \begin{pmatrix} X^1 \\ X^2 \end{pmatrix}^T \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \begin{pmatrix} Y^1 \\ Y^2 \end{pmatrix}$
- $\|\vec{X}\| := \sqrt{I_p(\vec{X}, \vec{X})}$ and $\measuredangle(\vec{X}, \vec{Y}) := \arccos\left(\frac{I_p(\vec{X},\vec{Y})}{\|\vec{X}\| \cdot \|\vec{Y}\|} \right)$
- $(g^{k\ell}) := (g_{ij})^{-1}$
- $\tilde{\vec{x}} = \vec{x} \circ F$ implies $\tilde{g}_{\alpha\beta} = \sum_{i,j=1}^2 \frac{\partial F^i}{\partial \tilde{u}^\alpha} \frac{\partial F^j}{\partial \tilde{u}^\beta} g_{ij}$
- $\vec{x}_{ij} := \frac{\partial^2 \vec{x}}{\partial u^i \, \partial u^j}$
- $\kappa_n := \langle \frac{d}{ds}\vec{T}, \vec{n} \rangle$ and $\kappa_g := \langle \frac{d}{ds}\vec{T}, \vec{S} \rangle$
- $L_{ij} := \langle \vec{x}_{ij}, \vec{n} \rangle$
- $\Gamma_{ij}^k := \sum_{\ell=1}^2 \langle \vec{x}_{ij}, \vec{x}_\ell \rangle \, g^{\ell k}$
- For $\vec{\alpha}$ unit-speed, $\vec{\alpha}'' = \kappa_n \, \vec{n} + \kappa_g \, \vec{S}$
- $\kappa_n = \sum_{i,j=1}^2 (\alpha_U^i)'(\alpha_U^j)' L_{ij}$ and $\kappa_g \, \vec{S} = \sum_{k=1}^2 \left((\alpha_U^k)'' + \sum_{i,j=1}^2 (\alpha_U^i)'(\alpha_U^j)' \Gamma_{ij}^k \right) \vec{x}_k$
- $D_{\vec{v}} f := (f \circ \vec{\alpha})'(0)$, where $\vec{\alpha}(0) = p$ and $\vec{\alpha}'(0) = \vec{v}$
- $\vec{n} = \nu \circ \vec{x}$
- $\mathcal{L}(\vec{v}) := -D_{\vec{v}} \nu$
- $\vec{n}_i := \frac{\partial \vec{n}}{\partial u^i}$
- $\vec{n}_j = -\sum_{i=1}^2 L_j^i \, \vec{x}_i$
- $(L_{ij}) = (g_{ij})(L_j^i)$, so $(L_j^i) = (g^{k\ell})(L_{ij})$

1. Let $\vec{\alpha}(t) = (r(t), z(t))$, $t \in (a, b)$, be a regular, C^∞, injective curve in the rz-plane, with $r(t) > 0$. Consider the simple surface

$$\vec{x}(t, \theta) := (r(t)\,\cos\theta, r(t)\,\sin\theta, z(t)), \quad t \in (a, b), \theta \in (-\pi, \pi).$$

(a) (6 points) Verify (using the definition) that $\vec{x}$ is a simple surface.
Hint: You'll need to compute $\vec{x}_1 \times \vec{x}_2$, which you can alternatively denote by $\vec{x}_t \times \vec{x}_\theta$.

Solution: Because $\vec{\alpha}(t)$, $\cos\theta$, and $\sin\theta$ are all C^∞, it follows that $\vec{x}$ is C^∞. Suppose we have $\vec{x}(t_0, \theta_0) = \vec{x}(t_1, \theta_1)$. Then

$$r(t_0)\,\cos\theta_0 = r(t_1)\,\cos\theta_1, \quad r(t_0)\,\sin\theta_0 = r(t_1)\,\sin\theta_1, \quad \text{and} \quad z(t_0) = z(t_1).$$

From the first two equations we see that

$$(r(t_0))^2 = (r(t_0)\,\cos\theta_0)^2 + (r(t_0)\,\sin\theta_0)^2 = (r(t_1)\,\cos\theta_1)^2 + (r(t_1)\,\sin\theta_1)^2 = (r(t_1))^2,$$

so $r(t_0) = r(t_1)$ and $z(t_0) = z(t_1)$. Since $\vec{\alpha}$ is injective, we must have $t_0 = t_1$. Finally, the first two equations above now reduce to

$$\cos\theta_0 = \cos\theta_1 \quad \text{and} \quad \sin\theta_0 = \sin\theta_1,$$

from which we see that $\theta_0 = \theta_1$. So $\vec{x}$ is injective. To check regularity, we compute

$$\vec{x}_1 = \langle \dot{r}(t)\,\cos\theta, \dot{r}(t)\,\sin\theta, \dot{z}(t)\rangle \quad \text{and} \quad \vec{x}_2 = \langle -r(t)\,\sin\theta, r(t)\,\cos\theta, 0\rangle.$$

Thus

$$\vec{x}_1 \times \vec{x}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dot{r}(t)\,\cos\theta & \dot{r}(t)\,\sin\theta & \dot{z}(t) \\ -r(t)\,\sin\theta & r(t)\,\cos\theta & 0 \end{vmatrix} = \langle -r\,\dot{z}\,\cos\theta, -r\,\dot{z}\,\sin\theta, r\,\dot{r}\rangle.$$

Notice that $\|\vec{x}_1 \times \vec{x}_2\|^2 = r^2(\dot{r}^2 + \dot{z}^2)$. Since $\vec{\alpha}$ is regular, $\dot{r}^2 + \dot{z}^2 > 0$, and we have $r > 0$ by assumption. So $\vec{x}_1 \times \vec{x}_2$ is never zero, and therefore $\vec{x}$ is regular.

(b) (4 points) Show that the matrix of metric coefficients for $\vec{x}$ is given by

$$(g_{ij}) = \begin{pmatrix} \dot{r}^2 + \dot{z}^2 & 0 \\ 0 & r^2 \end{pmatrix}.$$

Solution: We computed $\vec{x}_1$ and $\vec{x}_2$ above, from which we have

$$\begin{aligned} \langle \vec{x}_1, \vec{x}_1\rangle &= \dot{r}^2\,\cos^2\theta + \dot{r}^2\,\sin^2\theta + \dot{z}^2 = \dot{r}^2 + \dot{z}^2 \\ \langle \vec{x}_1, \vec{x}_2\rangle &= -r\,\dot{r}\,\cos\theta\,\sin\theta + r\,\dot{r}\,\cos\theta\,\sin\theta + 0 = 0 \\ \langle \vec{x}_2, \vec{x}_2\rangle &= r^2\,\sin^2\theta + r^2\,\cos^2\theta + 0 = r^2. \end{aligned}$$

Therefore

$$\boxed{(g_{ij}) = \begin{pmatrix} \dot{r}^2 + \dot{z}^2 & 0 \\ 0 & r^2 \end{pmatrix}.}$$

2. Let $\vec{x}\colon U \to \mathbb{R}^3$ be a simple surface with matrix of metric coefficients (g_{ij}).
   (a) (6 points) Prove that the Christoffel symbols Γ_{ij}^k satisfy

$$\Gamma_{ij}^k = \frac{1}{2}\sum_{\ell=1}^{2} g^{\ell k}\left(\frac{\partial g_{i\ell}}{\partial u^j} - \frac{\partial g_{ij}}{\partial u^\ell} + \frac{\partial g_{\ell j}}{\partial u^i}\right).$$

Hint: Unwind the right hand side of this equation and show that it comes out to equal the definition of Γ_{ij}^k given on the formula sheet.

Solution: We can apply the product rule to each of the three terms in parentheses:

$$\begin{aligned}\frac{\partial g_{i\ell}}{\partial u^j} &= \frac{\partial}{\partial u^j}\left(\langle \vec{x}_i, \vec{x}_\ell\rangle\right) = \langle \vec{x}_{ji}, \vec{x}_\ell\rangle + \langle \vec{x}_i, \vec{x}_{j\ell}\rangle \\ \frac{\partial g_{ij}}{\partial u^\ell} &= \frac{\partial}{\partial u^\ell}\left(\langle \vec{x}_i, \vec{x}_j\rangle\right) = \langle \vec{x}_{\ell i}, \vec{x}_j\rangle + \langle \vec{x}_i, \vec{x}_{\ell j}\rangle \\ \frac{\partial g_{\ell j}}{\partial u^i} &= \frac{\partial}{\partial u^i}\left(\langle \vec{x}_\ell, \vec{x}_j\rangle\right) = \langle \vec{x}_{i\ell}, \vec{x}_j\rangle + \langle \vec{x}_\ell, \vec{x}_{ij}\rangle.\end{aligned}$$

The parenthetical term thus reduces to

$$\begin{aligned}\frac{\partial g_{i\ell}}{\partial u^j} - \frac{\partial g_{ij}}{\partial u^\ell} + \frac{\partial g_{\ell j}}{\partial u^i} &= \left(\langle \vec{x}_{ji}, \vec{x}_\ell\rangle + \langle \vec{x}_i, \vec{x}_{j\ell}\rangle\right) - \left(\langle \vec{x}_{\ell i}, \vec{x}_j\rangle + \langle \vec{x}_i, \vec{x}_{\ell j}\rangle\right) + \left(\langle \vec{x}_{i\ell}, \vec{x}_j\rangle + \langle \vec{x}_\ell, \vec{x}_{ij}\rangle\right) \\ &= \left(\langle \vec{x}_{ji}, \vec{x}_\ell\rangle + \langle \vec{x}_\ell, \vec{x}_{ij}\rangle\right) + \left(\langle \vec{x}_i, \vec{x}_{j\ell}\rangle - \langle \vec{x}_i, \vec{x}_{\ell j}\rangle\right) + \left(\langle \vec{x}_{i\ell}, \vec{x}_j\rangle - \langle \vec{x}_{\ell i}, \vec{x}_j\rangle\right) \\ &= 2\langle \vec{x}_{ij}, \vec{x}_\ell\rangle + 0 + 0.\end{aligned}$$

This means that the right hand side of the desired equation is

$$\frac{1}{2}\sum_{\ell=1}^{2} g^{\ell k}\left(\frac{\partial g_{i\ell}}{\partial u^j} - \frac{\partial g_{ij}}{\partial u^\ell} + \frac{\partial g_{\ell j}}{\partial u^i}\right) = \frac{1}{2}\sum_{\ell=1}^{2} g^{\ell k}\, 2\langle \vec{x}_{ij}, \vec{x}_\ell\rangle = \sum_{\ell=1}^{2}\langle \vec{x}_{ij}, \vec{x}_\ell\rangle\, g^{\ell k}.$$

But this last expression is the definition of Γ_{ij}^k, so we're finished!

   (b) (4 points) Write a complete sentence or two explaining why we care about the above equation. There are two particular words — starting with the letters "g" and "i" — that must appear in your sentence(s).

Solution: This equation tells us that the Christoffel symbols are **intrinsic**, since they can be computed in terms of the coefficients g_{ij}. This is important to us because it allows us to prove that **geodesic** curvature is intrinsic.

3. Let

$$\vec{x}(t,\theta) := (r(t)\,\cos\theta, r(t)\,\sin\theta, z(t))$$

be the simple surface considered in Problem 1. The assumptions on $(r(t), z(t))$ are as before, except now we additionally **assume that $\dot{\mathbf{r}}^2 + \dot{\mathbf{z}}^2 = \mathbf{1}$.**

(a) (6 points) Compute the coefficients L_{ij} of the second fundamental form. Recall that $L_{ij} := \langle \vec{x}_{ij}, \vec{n}\rangle$.

Solution: We previously found that

$$\vec{x}_1 = \langle \dot{r}\,\cos\theta, \dot{r}\,\sin\theta, \dot{z}\rangle, \quad \vec{x}_2 = \langle -r\,\sin\theta, r\,\cos\theta, 0\rangle, \quad \text{and} \quad \vec{x}_1\times\vec{x}_2 = \langle -r\,\dot{z}\,\cos\theta, -r\,\dot{z}\,\sin\theta, r\,\dot{r}\rangle.$$

Since $\|\vec{x}_1 \times \vec{x}_2\| = r\sqrt{\dot{r}^2 + \dot{z}^2} = r$, we see that

$$\vec{n} = \langle -\dot{z}\,\cos\theta, -\dot{z}\,\sin\theta, \dot{r}\rangle.$$

Next, we compute the second derivatives:

$$\vec{x}_{11} = \langle \ddot{r}\,\cos\theta, \ddot{r}\,\sin\theta, \ddot{z}\rangle, \quad \vec{x}_{12} = \langle -\dot{r}\,\sin\theta, \dot{r}\,\cos\theta, 0\rangle, \quad \text{and} \quad \vec{x}_{22} = \langle -r\,\cos\theta, -r\,\sin\theta, 0\rangle.$$

Taking inner products yields:

$$\begin{aligned} L_{11} &= \langle \vec{x}_{11}, \vec{n}\rangle = -\ddot{r}\,\dot{z}\,\cos^2\theta - \ddot{r}\,\dot{z}\,\sin^2\theta + \dot{r}\,\ddot{z} = \dot{r}\,\ddot{z} - \ddot{r}\,\dot{z} \\ L_{12} &= \langle \vec{x}_{12}, \vec{n}\rangle = \dot{r}\,\dot{z}\cos\theta\,\sin\theta - \dot{r}\,\dot{z}\cos\theta\,\sin\theta + 0 = 0 \\ L_{22} &= r\,\dot{z}\cos^2\theta + r\,\dot{z}\sin^2\theta + 0 = r\,\dot{z}. \end{aligned}$$

In matrix form we have

$$\boxed{(L_{ij}) = \begin{pmatrix} \dot{r}\,\ddot{z} - \ddot{r}\,\dot{z} & 0 \\ 0 & r\,\dot{z} \end{pmatrix}}.$$

(b) (4 points) Use the formula $(L^i_j) = (g^{k\ell})(L_{ij})$ (as well as your previous work) to compute the matrix representation (L^i_j) of the Weingarten map.

Solution: We have the matrix (g_{ij}) from Problem 1(b). In fact, because we're assuming that $\dot{r}^2 + \dot{z}^2 = 1$, we can simplify that matrix to get

$$(g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \quad \Rightarrow \quad (g^{k\ell}) = \begin{pmatrix} 1 & 0 \\ 0 & r^{-2} \end{pmatrix}.$$

From this it follows that

$$(L^i_j) = (g^{k\ell})(L_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & r^{-2} \end{pmatrix} \begin{pmatrix} \dot{r}\,\ddot{z} - \ddot{r}\,\dot{z} & 0 \\ 0 & r\,\dot{z} \end{pmatrix} = \boxed{\begin{pmatrix} \dot{r}\,\ddot{z} - \ddot{r}\,\dot{z} & 0 \\ 0 & \dot{z}/r \end{pmatrix}}.$$

4. (a) (3 points) Given a unit-speed surface curve $\vec{\alpha}$ on a simple surface $\vec{x}$, we have a unit tangent vector $\vec{T}(s) := \vec{\alpha}'(s)$ and a unit normal vector $\vec{n}(s)$ given by the surface normal at the point $\vec{\alpha}(s)$. Explain how to define $\vec{S}(s)$ in terms of $\vec{T}(s)$ and $\vec{n}(s)$ so that

$$\{\vec{T}(s), \vec{S}(s), \vec{n}(s)\}$$

is the *Darboux frame.* Write a complete sentence explaining why *intrinsic normal* is a good name for this vector.

> **Solution:** Because $\vec{T}$ and $\vec{n}$ are orthonormal vectors, we get an orthonormal basis by setting $\vec{S} = \vec{n} \times \vec{T}$. Because ordered bases are *cyclically* ordered, this definition ensures that $\{\vec{T}, \vec{S}, \vec{n}\}$ is right-handed (reversing the order of $\vec{n}$ and $\vec{T}$ in the cross product would break this). We call this an *intrinsic normal* because it's a normal vector to our curve which is tangent to the surface (unlike the surface normal).

(b) (2 points) What coefficients belong in the blanks in the following equation?

$$\vec{x}_{ij} = \underline{\quad}\,\vec{n} + \sum_{k=1}^{2} \underline{\quad}\,\vec{x}_k.$$

No justification needed.

> **Solution:**
>
> $$\vec{x}_{ij} = \underline{L_{ij}}\,\vec{n} + \sum_{k=1}^{2} \underline{\Gamma_{ij}^{k}}\,\vec{x}_k.$$

(c) (2 points) Let $\vec{\alpha}$ be a surface curve on a simple surface $\vec{x}$, and let $\vec{V}$ be a vector field along $\vec{\alpha}$. That is, for each point $\vec{\alpha}(t)$, $\vec{V}(t)$ gives us a vector based at $\vec{\alpha}(t)$ which is tangent to $\vec{x}$. Use **complete sentences** to explain what it means for $\vec{V}$ to be *parallel along* $\vec{\alpha}$. You may write mathematical symbols if you like, but you will be graded on your **words**.

> **Solution:** When we say that $\vec{V}$ is parallel along $\vec{\alpha}$, we mean that as we move along $\vec{\alpha}$, $\vec{V}$ *does not appear to change from the perspective of an inhabitant of the surface.* The vector $\vec{V}(t)$ may be changing as a vector in $\mathbb{R}^3$, but only in ways that are necessary in order to remain tangent to the surface. Because an inhabitant of the surface is also making these changes, $\vec{V}(t)$ will not appear to be changing to the inhabitant.

(d) (3 points) Sketch a surface $\mathcal{S}$ and choose a point $p \in \mathcal{S}$ such that the Weingarten map $\mathcal{L}_p$ can be represented by the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Indicate the point p that you've chosen, and **draw the surface normal $\vec{n}$ at this point**. You don't have to justify your sketch, but you are welcome to add words to explain what you've drawn.

Solution: