Math 4441 Midterm 1

September 30, 2022

Name: ______________________ **gtID**: ______________________

Instructions. Read each question carefully and show all your work. Answers without justification will receive little to no credit. Writing your answers in a legible, well-organized manner will maximize your opportunities for partial credit.

This is a closed-note, closed-book exam, and you are expected to abide by the Georgia Tech Honor Challenge. Good luck!

Clearly label any extra papers you want graded.

By signing below, I certify that all work submitted on this exam is my own, and that I have neither given nor received any unauthorized help on this exam.

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Question	Points	Score
1	10	
2	10	
3	10	
4	10	
Total:	40	

Formulas

- Tangent line: $\vec{\ell}(\lambda) := \vec{\alpha}(t_0) + \lambda \vec{T}(t_0)$
- Arc length: $\int_a^b \|\vec{\alpha}'(t)\| \, dt$
- Magnitude: $\|\vec{v}\| := \sqrt{\vec{v} \cdot \vec{v}}$
- Angle: $\measuredangle(\vec{v}, \vec{w}) := \cos^{-1}\left(\dfrac{\vec{v} \cdot \vec{w}}{\|\vec{v}\| \cdot \|\vec{w}\|}\right)$
- Cauchy-Schwarz: $|\langle \vec{v}, \vec{w} \rangle| \leq \|\vec{v}\| \cdot \|\vec{w}\|$, with equality iff $\vec{v} \parallel \vec{w}$.
- Curvature: $\kappa(s) := \|\frac{d}{ds}\vec{T}(s)\|$
- For $\vec{\alpha}$ unit-speed, with $\kappa(s) \neq 0$, $\vec{B} := \vec{T} \times \vec{N}$.
- Torsion: $\tau(s) := -\langle \vec{B}'(s), \vec{N}(s) \rangle$
- For $\vec{\alpha}$ unit-speed, with $\kappa(s) \neq 0$:

$$\begin{pmatrix} \vec{T}'(s) \\ \vec{N}'(s) \\ \vec{B}'(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix} \begin{pmatrix} \vec{T}(s) \\ \vec{N}(s) \\ \vec{B}(s) \end{pmatrix}.$$

- For $\vec{\alpha}$ regular,

$$\kappa(t) = \frac{\|\vec{\alpha}'(t) \times \vec{\alpha}''(t)\|}{\|\vec{\alpha}'(t)\|^3} \quad \text{and} \quad \tau(t) = \frac{\langle \vec{\alpha}'(t) \times \vec{\alpha}''(t), \vec{\alpha}'''(t) \rangle}{\|\vec{\alpha}'(t) \times \vec{\alpha}''(t)\|^2}.$$

- For $\vec{\alpha}$ planar and unit-speed, $k(s) := \langle \frac{d}{ds}\vec{t}(s), \vec{n}(s) \rangle$.
- For $\vec{\alpha}$ planar and regular,

$$k(t) = \frac{\langle \vec{\alpha}''(t), J(\vec{\alpha}'(t)) \rangle}{\|\vec{\alpha}'(t)\|^3} = \frac{x'(t)y''(t) - x''(t)y'(t)}{((x'(t))^2 + (y'(t))^2)^{3/2}}.$$

- If $k(t) \neq 0$, $\vec{\varepsilon}(t) = \vec{\alpha}(t) + \frac{1}{k(t)}\vec{n}(t)$.
- $\int_{\vec{\alpha}} f \, dx + g \, dy := \int_a^b \langle f(\vec{\alpha}(t)), g(\vec{\alpha}(t)) \rangle \cdot \vec{\alpha}'(t) \, dt$
- If $\vec{\alpha}$ is regular, bounds a region $\mathcal{R} \subset \mathbb{R}^2$, and is oriented counter-clockwise, then

$$\oint_{\vec{\alpha}} f \, dx + g \, dy = \iint_{\mathcal{R}} \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dA.$$

- Angular rotation function: $\theta(s) := \theta_0 + \int_0^s k(u) \, du$.
- Rotation index: $i_{\vec{\alpha}} := \dfrac{\theta(L) - \theta(0)}{2\pi}$.

1. Let $\vec{\alpha}(t) = (e^t \cos t, e^t \sin t, 0)$, for $t \in (-\infty, \infty)$.

   (a) (3 points) Verify that $\vec{\alpha}$ is regular.

   Solution: We have

   $$\vec{\alpha}'(t) = (e^t \cos t - e^t \sin t, e^t \sin t + e^t \cos t, 0),$$

   so

   $$\begin{aligned} \|\vec{\alpha}'(t)\|^2 &= \langle \vec{\alpha}'(t), \vec{\alpha}'(t) \rangle \\ &= (e^t \cos t - e^t \sin t)^2 + (e^t \sin t + e^t \cos t)^2 \\ &= 2e^{2t}. \end{aligned}$$

   It follows that $\|\vec{\alpha}'(t)\| = \sqrt{2}e^t > 0$, for all $t \in (-\infty, \infty)$. So $\vec{\alpha}$ is regular.

   (b) (3 points) Let $\vec{\beta}(u) = (u \cos(\ln u), u \sin(\ln u), 0)$, for $u \in (0, \infty)$. Is $\vec{\beta}$ a reparametrization of $\vec{\alpha}$? Why or why not?
   Note: Remember that $\vec{\beta}$ is a reparametrization of $\vec{\alpha}$ *if we can write $\vec{\beta} = \vec{\alpha} \circ g$, for some C^k reparametrization g. For this problem, take $k = 1$.*

   Solution: Notice that $\vec{\beta}(u) = \vec{\alpha}(\ln u)$, so this is a matter of determining whether or not the function $g\colon (0, \infty) \to (-\infty, \infty)$ defined by $g(u) := \ln u$ is a C^1-reparametrization. Indeed, g is a bijection, with $g^{-1}(t) = e^t$. Since $g'(u) = 1/u$ and $(g^{-1})'(t) = e^t$ are continuous on their respective domains, g and g^{-1} are C^1, so g is a reparametrization.

   (c) (4 points) Prove that the angle between $\vec{\alpha}$ and $\vec{T}$ is constant.
   Note: This part is treating $\vec{\alpha}$ as a vector based at $\vec{0}$, and does not use part (b).

   Solution: The angle between $\vec{\alpha}$ and $\vec{T}$ is the same as the angle between $\vec{\alpha}$ and $\vec{\alpha}'$, since angles don't care about magnitudes. In particular, we're concerned about the angle given by

   $$\cos^{-1}\left(\frac{\langle \vec{\alpha}(t), \vec{\alpha}'(t) \rangle}{\|\vec{\alpha}(t)\| \, \|\vec{\alpha}'(t)\|}\right).$$

   We've already computed $\vec{\alpha}'(t)$ and $\|\vec{\alpha}'(t)\|$. Notice that

   $$\langle \vec{\alpha}(t), \vec{\alpha}'(t) \rangle = e^{2t} \cos^2 t - e^{2t} \cos t \sin t + e^{2t} \sin^2 t + e^{2t} \cos t \sin t = e^{2t}$$

   and

   $$\|\vec{\alpha}(t)\| = \sqrt{e^{2t} \cos^2 t + e^{2t} \sin^2 t} = \sqrt{e^{2t}} = e^t.$$

   We previously computed that $\|\vec{\alpha}'(t)\| = \sqrt{2}e^t$, so

   $$\cos^{-1}\left(\frac{\langle \vec{\alpha}(t), \vec{\alpha}'(t) \rangle}{\|\vec{\alpha}(t)\| \, \|\vec{\alpha}'(t)\|}\right) = \cos^{-1}\left(\frac{e^{2t}}{\sqrt{2}e^{2t}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}.$$

   Notice that this angle is independent of t, as desired. (You didn't have to compute the exact angle to complete this problem, provided you showed that the angle is constant.)

2. Let $\vec{\alpha}(s)$ be an arbitrary unit-speed curve in $\mathbb{R}^3$ with nonvanishing curvature $\kappa(s) > 0$. The goal of this problem is to identify the **Darboux vector** for $\vec{\alpha}$, which is the unique vector $\vec{\omega}(s)$ satisfying

$$\vec{T}' = \vec{\omega} \times \vec{T}, \quad \vec{N}' = \vec{\omega} \times \vec{N}, \quad \text{and} \quad \vec{B}' = \vec{\omega} \times \vec{B}, \tag{1}$$

for all s in the domain of $\vec{\alpha}$.

(a) (6 points) Derive a formula for $\vec{\omega}$ in terms of the Frenet frame. That is, identify functions $a(s)$, $b(s)$, and $c(s)$ for which

$$\vec{\omega}(s) = a(s)\,\vec{T}(s) + b(s)\,\vec{N}(s) + c(s)\,\vec{B}(s).$$

Show the steps of your derivation, and **box your formula for $\vec{\omega}$.**
Disclaimer: In retrospect, I wish I'd given more of a hint for how to start this one, because our proof of the Frenet-Serret equations makes it really natural to write an expansion of $\vec{\omega}$ using inner products and then try to find those inner products. That approach works, but is messier than intended. I'll keep this in mind if/when a curve is set at the end of the term.

Solution: The equation $\vec{T}' = \vec{\omega} \times \vec{T}$ tells us that

$$\begin{aligned} \vec{T}'(s) &= a(s)\,(\vec{T}(s) \times \vec{T}(s)) + b(s)\,(\vec{N}(s) \times \vec{T}(s)) + c(s)\,(\vec{B}(s) \times \vec{T}(s)) \\ &= -b(s)\,\vec{B}(s) + c(s)\,\vec{N}(s). \end{aligned}$$

On the other hand, the Frenet-Serret equations tell us that $\vec{T}'(s) = \kappa(s)\,\vec{N}(s)$, so this suggests that we take $c(s) = \kappa(s)$ and $b(s) = 0$. From the equation $\vec{N}' = \vec{\omega} \times \vec{N}$ we have

$$\begin{aligned} \vec{N}'(s) &= a(s)\,(\vec{T}(s) \times \vec{N}(s)) + b(s)\,(\vec{N}(s) \times \vec{N}(s)) + c(s)\,(\vec{B}(s) \times \vec{N}(s)) \\ &= a(s)\,\vec{B}(s) - c(s)\,\vec{T}(s). \end{aligned}$$

This time the Frenet-Serret equations give us $\vec{N}'(s) = -\kappa(s)\,\vec{T}(s) + \tau(s)\,\vec{B}(s)$, again suggesting that $c(s) = \kappa(s)$, and also tells us to set $a(s) = \tau(s)$. So we propose

$$\boxed{\vec{\omega}(s) := \tau(s)\,\vec{T}(s) + \kappa(s)\,\vec{B}(s)}$$

as the formula for $\vec{\omega}$.

(b) (4 points) Carefully check that the vector $\vec{\omega}$ defined by your formula satisfies the three equations listed in Equation (1).

Solution: We take $\vec{\omega}$ as defined above. Then

$$\begin{aligned} \vec{\omega} \times \vec{T} &= \tau\,(\vec{T} \times \vec{T}) + \kappa\,(\vec{B} \times \vec{T}) = \kappa\,\vec{N} = \vec{T}' \\ \vec{\omega} \times \vec{N} &= \tau\,(\vec{T} \times \vec{N}) + \kappa\,(\vec{B} \times \vec{N}) = -\kappa\,\vec{T} + \tau\,\vec{B} = \vec{N}' \\ \vec{\omega} \times \vec{B} &= \tau\,(\vec{T} \times \vec{B}) + \kappa\,(\vec{B} \times \vec{B}) = -\tau\,\vec{N} = \vec{B}', \end{aligned}$$

as desired. The last line of each equation uses the Frenet-Serret equations.

Hint: Use the Frenet-Serret equations throughout.

3. Let $\vec{\alpha}$ be a regular, planar, simple, closed, unit-speed curve. Suppose there is a constant c such that the planar curvature $k(s)$ of $\vec{\alpha}$ satisfies

$$0 \leq k(s) \leq c, \tag{2}$$

for all s.

(a) (4 points) The inequality (2) ensures that $\vec{\alpha}$ has an important global property. What is this property? (Write a complete sentence which cites a theorem we proved.)

Solution: The inequality tells us that the planar curvature of $\vec{\alpha}$ has constant sign. Because $\vec{\alpha}$ is simple and closed, the theorem we proved in Activity 4 tells us that $\vec{\alpha}$ is **convex**.

(b) (6 points) Prove that length$(\vec{\alpha}) \geq 2\pi/c$.
Hint: Use the rotation index theorem.

Solution: Let L denote the length of $\vec{\alpha}$. Because $\vec{\alpha}$ is simple and closed, the rotation index theorem tells us that

$$\int_{\vec{\alpha}} k(s)\, ds = \pm 2\pi.$$

Because $k(s) \geq 0$, the right hand side must in fact be positive 2π. At the same time,

$$\int_{\vec{\alpha}} k(s)\, ds = \int_0^L k(s)\, ds \leq \int_0^L c\, ds = cL.$$

So we see that $cL \geq 2\pi$, and thus $L \geq 2\pi/c$.

4. (a) (3 points) Consider the following maps $f, g\colon \mathbb{R}^3 \to \mathbb{R}^3$:

$$f(\vec{x}) := \begin{pmatrix} 0 & 0 & 1 \\ 2 & 0 & 0 \\ 0 & 1/2 & 0 \end{pmatrix} \vec{x} + \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}, \qquad g(\vec{x}) := \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix} \vec{x} + \begin{pmatrix} 5 \\ 1 \\ -7 \end{pmatrix}.$$

Exactly one of these is an isometry; which one is it? Justify your response.

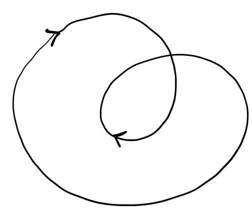
Solution: Each of these maps has the form $\vec{x} \mapsto A\vec{x} + \vec{c}$ for some constant matrix A and constant vector $\vec{c}$. But to be an isometry, A must be an orthogonal matrix, which is equivalent to the columns of A forming an orthonormal basis for $\mathbb{R}^3$. This is true of g, but not f. So g is the isometry.

(b) (2 points) Consider the ellipse $x^2+9y^2 = 9$. Which point(s) on the ellipse has the smallest osculating circle? Which point(s) has the largest osculating circle? (No computation required, but explain your reasoning, possibly supported with a figure.)

Solution: The points $(\pm 3, 0)$ have the largest curvature, and thus the smallest osculating circles; the points $(0, \pm 1)$ have the smallest curvature, and thus the largest osculating circles.

(c) (2 points) Draw an oriented, closed, planar curve with rotation index -2. No justification needed.

Solution:



(d) (3 points) Let $\vec{\alpha}(t) = (x(t), y(t))$ be a closed, regular, planar curve, oriented counterclockwise. Use Green's theorem to prove that

$$A = \oint_{\vec{\alpha}} x\,dy,$$

where A is the area enclosed by $\vec{\alpha}$.

Solution: According to Green's theorem (as stated on the formula sheet above),

$$\oint_{\vec{\alpha}} x\,dy = \iint_{\mathcal{R}} \left(\frac{\partial}{\partial x}(x) - 0 \right) dA = \iint_{\mathcal{R}} 1\,dA = A.$$