6 Homework 6

Instructions

Complete the following exercises and upload your work to Gradescope by **11:59 pm on December 12**.

Solutions to single-starred exercises do not need to be submitted, but you should know how to do them. Of the solutions you submit, some will be checked carefully while others are graded for submission only.

Solutions to double-starred exercises will become optional if 85% of enrolled students complete the CIOS evaluation by 11:59pm on December 9.

Be sure to **acknowledge your collaborators** and any resources you reference!

Acknowledgment: Several of the problems below are adapted from do Carmo's *Differential Geometry of Curves and Surfaces*, Millman & Parker's *Elements of Differential Geometry*, and Pressley's *Elementary Differential Geometry*.

1.** For any simple surface $\vec{x}$, prove that $(L_{ij}) = (g_{ij})(L^i_j)$.
Hint: See the last few slides from November 14.

2. Let $\vec{x}$ be a simple surface in $\mathbb{R}^3$, and suppose that the Weingarten map $\mathcal{L}_p$ is 0, for all points p in $\operatorname{im}(\vec{x})$. Show that $\operatorname{im}(\vec{x})$ is contained in a plane.
Hint: Start by arguing that $\nu\colon \operatorname{im}(\vec{x}) \to \mathbb{R}^3$ is a constant map. Then fix a point p in $\operatorname{im}(\vec{x})$ and argue that $\operatorname{im}(\vec{x})$ lies in the plane through p and perpendicular to the fixed output $\vec{n}_0$ of ν.

3.* Compute the Gaussian and mean curvatures of the **helicoid** $\vec{x}\colon (-\infty,\infty)\times(0,\infty)\to\mathbb{R}^3$ defined by

$$\vec{x}(u^1,u^2) := (u^2\cos u^1, u^2\sin u^1, \lambda u^1),$$

where $\lambda > 0$ is a constant.

4. Show that the Weingarten map $\mathcal{L}$ of a surface satisfies the equation

$$\mathcal{L}^2 - 2H\,\mathcal{L} + K\,\mathrm{id} = 0,$$

where H and K are the mean and Gaussian curvatures of the surface, respectively, and id is the identity map on the tangent space.
Hint: Recall the Cayley-Hamilton theorem from linear algebra.

5.* Prove that the mean and Gaussian curvatures of a surface satisfy $H^2 \geq K$. When does equality hold?

6.** Show that there is no simple surface $\vec{x}$ whose first and second fundamental forms are given by

$$(g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & \cos^2 u^1 \end{pmatrix} \quad\text{and}\quad (L_{ij}) = \begin{pmatrix} \cos^2 u^1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Hint: Show that the second Codazzi-Mainardi equation is violated. Use the intrinsic formula to compute the Christoffel symbols.

7.* Suppose we have a simple surface $\vec{x}\colon (-\infty,\infty)\times(0,\infty)\to\mathbb{R}^3$ whose first fundamental form is given by

$$(g_{ij}) = \begin{pmatrix} (u^2)^{-2} & 0 \\ 0 & (u^2)^{-2} \end{pmatrix},$$

and whose second fundamental form satisfies $L_{12} = 0$.

(a) Prove that L_{22} is independent of u^1.
Hint: Use the Codazzi-Mainardi equations.

(b) Prove that $K = -1$.
Hint: Use the Gauss equations.

(c) Prove that $L_{11}L_{22} = -(u^2)^{-4}$.
Hint: Recall that $K = \det(L_{ij})/\det(g_{ij})$.

(d) Prove that $L_{11}\,(u^2)^5\,\frac{dL_{11}}{du^2} = 1-(L_{11})^2\,(u^2)^4$.

(e) Solve the preceding differential equation for L_{11} and deduce that $\vec{x}$ cannot exist. (Namely, the domain of $\vec{x}$ cannot be all of $(-\infty,\infty)\times(0,\infty)$.)

8.** Consider the simple surfaces $\vec{x}, \tilde{\vec{x}} : (0,\infty)\times(-\infty,\infty) \to \mathbb{R}^3$ defined by

$$\vec{x}(u^1,u^2) = (u^1\,\cos u^2, u^1\,\sin u^2, \ln u^1) \quad \text{and} \quad \tilde{\vec{x}}(u^1,u^2) = (u^1\,\cos u^2, u^1\,\sin u^2, u^2)$$

Verify that $K(u^1,u^2) = \tilde{K}(u^1,u^2)$ for all (u^1,u^2), but that $(g_{ij}) \neq (\tilde{g}_{ij})$. This suggests that, while Gaussian curvature is preserved by isometry, it cannot detect isometries.

9.** Suppose a simple surface $\vec{x}$ has $g_{11} \equiv 1$ and $g_{12} \equiv 0$. Imitate our proof of the *Theorema Egregium* to prove that

$$\frac{\partial^2}{\partial(u^1)^2}(\sqrt{g_{22}}) + K\,\sqrt{g_{22}} = 0.$$

Hint: As in our proof, you'll need to consider the equality of some order-3 mixed partials.

10. Show that if a simple surface $\vec{x}$ has Gaussian curvature everywhere non-positive, then there are no simple, closed geodesics on $\vec{x}$. Why does this not contradict the fact that the parallels of a right circular cylinder are geodesics?
Hint: Use the Gauss-Bonnet theorem for surfaces with boundary.

11. Show that if $\mathcal{S}$ is a compact surface without boundary whose Gaussian curvature is everywhere positive, then $\mathcal{S}$ is diffeomorphic to a sphere. Is the converse of this statement true?

12.* Let $\mathcal{S} \subset \mathbb{R}^3$ be a compact surface without boundary whose Gaussian curvature is everywhere positive. Show that if $\vec{\alpha}$ and $\vec{\beta}$ are two simple, closed geodesics on $\mathcal{S}$, then they must intersect each other.
Hint: If the geodesics do not intersect, then they form the boundary of a surface $\mathcal{S}' \subset \mathcal{S}$ diffeomorphic to a cylinder.