5 Homework 5

Instructions

Complete the following exercises and upload your work to Gradescope by **11:59 pm on November 14**.

Solutions to the starred exercises do not need to be submitted, but you should know how to do them. Of the solutions you submit, some will be checked carefully while others are graded for submission only.

Be sure to **acknowledge your collaborators** and any resources you reference!

Acknowledgment: Several of the problems below are adapted from Millman & Parker's *Elements of Differential Geometry*.

1. Let $\vec{\alpha} = \vec{x} \circ \vec{\alpha}_U$ be a regular surface curve on a simple surface. Show that $D_{\vec{\alpha}'(s)}\vec{\alpha}' = \alpha''(s)$, for every s in the domain of $\vec{\alpha}$.
Hint: Working from the definition of directional derivatives, it will be helpful to write down a curve with the same image as $\vec{\alpha}$.

2. Let $\vec{\alpha} = \vec{x} \circ \vec{\alpha}_U$ be a regular surface curve on a simple surface. Show that $\vec{\alpha}$ is a geodesic if and only if $\nabla_{\vec{\alpha}'(s)}\vec{\alpha}' \equiv \vec{0}$.
Hint: This should follow pretty quickly from Problem 1, plus a fact about the acceleration of geodesics.

3.* Let $\vec{\alpha}(s) = (x(s), y(s))$, $s \in (a, b)$ be a simple, unit-speed plane curve. Define a simple surface $\vec{x} : (a, b) \times \mathbb{R} \to \mathbb{R}^3$ by
$$\vec{x}(s, t) := (x(s), y(s), t),$$
and consider the surface curve $\vec{\gamma}(u) := \vec{x}(u, u \tan\psi)$, for some fixed constant ψ (such that $\tan\psi$ makes sense). Prove that $\vec{\gamma}$ is a geodesic, up to arclength reparametrization.

4. Let $\mathcal{S}$ be the surface given by $x^2 + y^2 - z^2 = 1$, and let $p \in \mathcal{S}$ be the point $(1, 0, 0)$. Sketch three distinct geodesics on $\mathcal{S}$ passing through p, and explain in complete sentences why you know that they are geodesics. (No computations needed, but take care with your sketch and sentences.)
Hint: Use Clairaut's relation.

5. Let $\mathcal{S}$ be the cylinder $x^2 + y^2 = 1$. Show that if $p, q \in \mathcal{S}$ are distinct points on the cylinder, then there are either exactly two geodesics with endpoints p and q, or there are infinitely many such geodesics. Which pairs of points have exactly two geodesics between them?
Hint: Use Clairaut's relation.

6.* Give an example of a pair of points on a simple surface with no geodesic connecting them. (You should write some sentences and draw a picture, but don't necessarily have to do any computations.)

7. We showed in the last homework that the geodesics of a sphere are the great circles. Since geodesics are supposed to be for surfaces what lines are for the plane, determine which of the following sentences is true when we replace "line" with "geodesic." For sentences which become false, give an example or explanation.
   (a) There is a line passing through any two distinct points.
   (b) There is a *unique* line passing through any two distinct points.
   (c) Any two distinct lines intersect in at most one point.
   (d) Given a line ℓ and a point p not on ℓ, there is another line ℓ' which passes through p and does not intersect ℓ.
   (e) Any line can be continued indefinitely.
   (f) A line connecting two points gives the shortest path between the points.

(g) The shortest path between two points is a line.

8. Let $\vec{X}_N$ denote the tangential component of the normal vector $\vec{N}$ of a unit-speed surface curve $\vec{\alpha} = \vec{x} \circ \vec{\alpha}_U$. That is,

$$\vec{X}_{\vec{N}} := \vec{N} - \langle \vec{N}, \vec{n} \rangle \, \vec{n},$$

where $\vec{n}$ is the unit surface normal for $\vec{x}$. Prove that the following are equivalent:

(a) $\vec{X}_{\vec{N}} \equiv \vec{0}$;

(b) $\vec{\alpha}$ is a geodesic;

(c)* $\vec{X}_{\vec{N}}$ is parallel as a vector field along $\vec{\alpha}$, in the sense that $\nabla_{\vec{\alpha}'(s)} \vec{X}_{\vec{N}} \equiv 0$.

9. Consider the simple surface $\vec{x} : (-\pi, \pi) \times \mathbb{R} \to \mathbb{R}^3$ defined by

$$\vec{x}(u^1, u^2) := (\cos u^1, \sin u^1, u^1 + u^2),$$

whose image is the cylinder $x^2 + y^2 = 1$. Compute the matrix representation (L^i_j) of the Weingarten map $\mathcal{L}$ in the basis $\{\vec{x}_1, \vec{x}_2\}$. (This is an example where $L^i_j \neq L^j_i$.)

10.* Consider the familiar surface of revolution

$$\vec{x}(t, \theta) := (r(t) \cos \theta, r(t) \sin \theta, z(t)),$$

where, as usual, $r(t) > 0$. Assume that $\dot{r}^2 + \dot{z}^2 = 1$.

(a) Compute the matrix (L^i_j).

(b) Verify that $(L_{ij}) = (g_{ij})(L^i_j)$, where (g_{ij}) is the matrix of metric coefficients and (L_{ij}) is the matrix of coefficients of the second fundamental form, each computed in previous assignments.

Hint: In addition to using the assumption that $\dot{r}^2 + \dot{z}^2 = 1$*, the derivative of this equation will also be helpful.*