4 Homework 4

Instructions

Complete the following exercises and upload your work to Gradescope by **11:59 pm on October 28**.

Solutions to the starred exercises do not need to be submitted, but you should know how to do them. Of the solutions you submit, some will be checked carefully while others are graded for submission only.

Be sure to **acknowledge your collaborators** and any resources you reference!

Acknowledgment: Several of the problems below are adapted from Millman & Parker's *Elements of Differential Geometry*.

1. Suppose there were[1] a simple surface $\vec{x}\colon (-\infty,\infty)\times(0,\infty)\to\mathbb{R}^3$ whose matrix of metric coefficients is given by

$$(g_{ij}) = \begin{pmatrix} (u^2)^{-2} & 0 \\ 0 & (u^2)^{-2} \end{pmatrix}.$$

We'll denote the domain of $\vec{x}$ by $U = (-\infty,\infty)\times(0,\infty)$. The purpose of this problem is to see that we can do some computations working from the matrix of metric coefficients alone, without a formula for $\vec{x}$.

   (a) For any $c \in (0,\infty)$, give a unit-speed parametrization of the surface curve $u^2 = c$. That is, we can define a surface curve $\vec{\alpha}\colon (-\infty,\infty)\to\mathbb{R}^3$ by $\vec{\alpha}(t) = \vec{x}(t,c)$, and you should reparametrize $\vec{\alpha}$ by arc length.

   (b) Repeat part 1a, but for the u^2-curves. That is, for any $h \in (-\infty,\infty)$, give a unit-speed parametrization of $\vec{\beta}(t) = \vec{x}(h,t)$, $t \in (0,\infty)$.
   Hint: When you go to build the arclength function (by integrating $\|\vec{\beta}'(u)\|$), you can't use 0 as the lower bound, since $t = 0$ isn't in our domain. Use $t = 1$ as your lower bound instead.

   (c) Use equation (30) on page 58 of the course text to show that the Christoffel symbols Γ^k_{ij} of $\vec{x}$ satisfy

$$\Gamma^2_{11} = \frac{1}{u^2}, \quad \Gamma^1_{12} = \Gamma^1_{21} = \Gamma^2_{22} = -\frac{1}{u^2},$$

   with all others zero.

   (d) Using your unit-speed parametrizations, compute the geodesic curvatures of the u^1- and u^2-curves. You should find that one of these types of curves gives geodesics, while the other does not. Both have constant curvature. Sketch a few u^1- and u^2-curves in U, and indicate which are geodesics and which have nonzero curvature.
   Hint: Equation (31) of the course text will help here. It looks messy, but keep in mind that your downstairs curves are pretty simple.

   (e) Why does this problem not ask you to find the normal curvatures of these curves?
   Hint: It's not just because I'm nice.

2.* This problem is a continuation of problem 1; let all notation be as given there.

   (a) Consider a coordinate transformation $f : (-\infty,\infty)\times(0,\infty)\to(-\infty,\infty)\times(0,\infty)$ defined by

$$f(u^1,u^2) := \left(\frac{(u^1)^2+(u^2)^2-1}{(u^1+1)^2+(u^2)^2}, \frac{2u^2}{(u^1+1)^2+(u^2)^2}\right),$$

[1] This footnote previously had a mistake: it said that a surface such as this could not exist. It should have said that such a surface cannot be *complete*.

and then define a new simple surface $\tilde{\vec{x}} := \vec{x} \circ f$. (You don't need to check that f is a coordinate transformation; also, notice that the domain of $\tilde{\vec{x}}$ is the same as that of $\vec{x}$.) Verify that $(\tilde{g}_{\alpha\beta}) = (g_{ij})$, so that it makes sense to call f an **isometry**.
Hint: Use the formulas from the October 10 notes.

(b) Prove that if $\vec{\alpha} = \vec{x} \circ \vec{\alpha}_U$ is a geodesic of $\vec{x}$, then $\vec{\alpha}_f = \vec{x} \circ (f \circ \vec{\alpha}_U)$ is a geodesic of $\vec{x}$, where f is as in part 2a.
Hint: Use our formula for κ_g in terms of g and the downstairs curve.

(c) Based on part 2b, we have a strategy for finding new geodesics of $\vec{x}$: start with one geodesic of $\vec{x}$ (say, the u^2-curve with $u^1 = 0$) and then apply isometries of the domain. Use parts 2a and 2b to show that the curve $\vec{x}(\cos t, \sin t)$, $t \in (0, \pi)$ is a geodesic.
Note: The given parametrization of this curve is not unit-speed, and won't arise naturally in your computation. You're really just trying to show that the upper half of the unit circle is the "downstairs curve" of a geodesic in $\vec{x}$.

3. Let $\vec{x} : (a, b) \times (-\pi, \pi) \to \mathbb{R}^3$ be a surface of revolution

$$\vec{x}(t, \theta) := (r(t)\cos\theta, r(t)\sin\theta, z(t)),$$

as in problem 1 of homework 3.

(a) Show that the matrix (L_{ij}) is given by

$$\frac{1}{\sqrt{\dot{r}^2 + \dot{z}^2}} \begin{pmatrix} \dot{r}\ddot{z} - \dot{z}\ddot{r} & 0 \\ 0 & r\dot{z} \end{pmatrix}.$$

(b) Prove that $\det(L_{ij}) \equiv 0$ if and only if each meridian is a straight line.

4.* Let $\vec{x} : U \to \mathbb{R}^3$ be a simple surface, $f : \tilde{U} \to U$ a coordinate transformation, and $\tilde{\vec{x}} = \vec{x} \circ f$ the resulting reparametrization of $\vec{x}$. Show that

$$L_{ij} = \frac{\det(df)}{|\det(df)|} \sum_{\alpha,\beta=1}^{2} \tilde{L}_{\alpha\beta} \frac{\partial f^\alpha}{\partial \tilde{u}^i} \frac{\partial f^\beta}{\partial \tilde{u}^j},$$

for each i, j, where $\tilde{L}_{\alpha\beta}$ are the coefficients of the second fundamental form of $\tilde{\vec{x}}$.
Hint: Your solution should be similar to our work in the October 10 notes.

5. Prove that if $\vec{\alpha}$ is a surface curve on a plane in $\mathbb{R}^3$, then $\kappa_g = \kappa$.
Hint: Refer to exercise 7.4 of activity 7. You can either use the function $\vec{x}$ given there or, more simply, use the result of that exercise plus the fact that $\kappa^2 = \kappa_g^2 + \kappa_n^2$.

6. Consider the sphere $\vec{x} : (0, \pi) \times (-\pi, \pi) \to \mathbb{R}^3$ defined by

$$\vec{x}(u^1, u^2) := (R \sin u^1 \cos u^2, R \sin u^1 \sin u^2, R \cos u^1),$$

for some fixed $R > 0$. You may recall[2] that the unit surface normal is given by

$$\vec{n}(u^1, u^2) := (\sin u^1 \cos u^2, \sin u^1 \sin u^2, \cos u^1).$$

(a) Find the geodesic curvature of the circle of latitude given by $u^1 = c$, for some fixed constant c.

(b) Prove that the normal curvature of any curve on the sphere is constant.
Hint: For this, it will be useful to notice that $\vec{n}$ is parallel to $\vec{x}$, and thus to a surface curve $\vec{\alpha}$, if we treat $\vec{\alpha}$ as a vector.

[2]Actually, the roles of u^1 and u^2 here are swapped from the version we're used to seeing. That's because the previous version had an *inward*-pointing normal vector, but it's more conventional to use an *outward*-pointing normal vector.

7. Show that a meridian of a surface of revolution is a geodesic without solving any differential equations. *Hint: The acceleration vector of a unit-speed geodesic has a nice relationship with the tangent plane.*

8. Show that any geodesic on a sphere is a great circle. *Hint: Use part (b) of problem 6, as well as the fact that $\kappa^2 = \kappa_g^2 + \kappa_n^2$.*

9. Suppose that a surface curve $\vec{\alpha}$ on a simple surface $\vec{x}$ is a straight line. Prove that $\vec{\alpha}$ is a geodesic. *Hint: Keep using a relation we've hinted a couple of times already.*

10. Suppose $\vec{x}$ is a simple surface whose metric coefficients satisfy $g_{11} \equiv 1$ and $g_{12} \equiv 0$. Prove that the u^1-curves are geodesics. We call such a simple surface a **geodesic coordinate patch**. *Hint: Differentiate the equation $g_{11} \equiv 1$ with respect to each of u^1 and u^2, and differentiate $g_{12} \equiv 0$ with respect to u^1, in order to learn things about $\vec{x}_{11}$.*

11.* Let $\vec{\alpha}(t)$ be a geodesic in a simple surface which we do not assume is parametrized by arclength. Prove that

$$\frac{d^2\alpha^i}{dt^2} + \sum_{j,k=1}^{2} \Gamma^i_{jk} \frac{d\alpha^j}{dt} \frac{d\alpha^k}{dt} = -\frac{d\alpha^i}{dt} \frac{d^2 t}{ds^2} \left(\frac{ds}{dt}\right)^2$$

for $i = 1, 2$. Here s is the arclength parameter for $\vec{\alpha}$, and we're treating s as a function of t or t as a function of s as needed.