3 Homework 3

Instructions

Complete the following exercises and upload your work to Gradescope by **11:59 pm on October 14**.

Solutions to the starred exercises do not need to be submitted, but you should know how to do them. Of the solutions you submit, some will be checked carefully while others are graded for submission only.

Be sure to **acknowledge your collaborators** and any resources you reference!

Acknowledgment: Several of the problems below are adapted from Millman & Parker's *Elements of Differential Geometry*, Shifrin's *Differential Geometry*, and do Carmo's *Differential Geometry of Curves and Surfaces*.

1. Let $\vec{\alpha}(t) = (r(t), z(t))$, $t \in (a,b)$ be a curve in the rz-plane, with $r(t) > 0$. Rotating this curve about the z-axis (in three dimensions) produces a *surface of revolution*. We can parametrize this surface[1] as

$$\vec{x}(t,\theta) := (r(t)\cos\theta, r(t)\sin\theta, z(t)), \quad t \in (a,b), \theta \in (-\pi,\pi).$$

   Prove that $\vec{x}$ is a simple surface, provided $\vec{\alpha}$ is regular and injective.
   Hint: Compute $\vec{x}_1 \times \vec{x}_2$, which you can alternatively denote by $\vec{x}_t \times \vec{x}_\theta$.

2.* Consider the surface $\vec{x}$ defined by

$$\vec{x}(\theta,v) := (\cos\theta, \sin\theta, 0) + v\,(\sin\tfrac{\theta}{2}\cos\theta, \sin\tfrac{\theta}{2}\sin\theta, \cos\tfrac{\theta}{2}), \quad \theta \in (-\pi,\pi), v \in (-\tfrac{1}{2}, \tfrac{1}{2}).$$

   This gives the *Möbius band*. Compute $\vec{n}(\theta,0)$ and show that

$$\lim_{\theta\to-\pi} \vec{x}(\theta,0) = -\lim_{\theta\to\pi} \vec{x}(\theta,0), \qquad \text{but} \qquad \lim_{\theta\to-\pi} \vec{n}(\theta,0) = -\lim_{\theta\to\pi} \vec{n}(\theta,0)$$

   (It might be worthwhile to plot this in *Mathematica*.)

3. Let $\vec{x}\colon (0,\infty) \times (-\pi,\pi) \to \mathbb{R}^3$ be defined by

$$\vec{x}(r,\theta) := (r\cos\theta, r\sin\theta, 0).$$

   (a) What surface in $\mathbb{R}^3$ is given by the image of $\vec{x}$?
   (b) Compute the matrix of metric coefficients for $\vec{x}$.

4. Let $\vec{x}$ be the surface of revolution defined in Exercise 1. Show that the matrix of metric coefficients is given by

$$(g_{ij}) = \begin{pmatrix} \dot{r}^2 + \dot{z}^2 & 0 \\ 0 & r^2 \end{pmatrix}.$$

5.* Let $\vec{x}\colon \mathbb{R}^2 \to \mathbb{R}^3$ be the parametrization of S^2 you computed in Exercise 5.3 of activity 5. (The parametrization corresponding to north pole stereographic projection.) Compute the matrix of metric coefficients of $\vec{x}$.

6. Fix some $m > 0$ and consider the simple surface

$$\vec{x}(u^1,u^2) := (\cos u^1, \sin u^1, u^1 + m u^2), \quad u^1 \in (-\infty,\infty), \ u^2 \in (0, m/2\pi).$$

   (a) Describe the surface given by $\mathrm{im}(\vec{x})$.

[1] Technically we won't get the whole surface, since we're excluding $\theta = \pi$.

(b) Compute (g^{kl}).

7. A simple surface $\vec{x}\colon U \to \mathbb{R}^3$ is called an *orthogonal net* if the u^1-curves meet the u^2-curves at right angles (i.e., if $\vec{x}_1 \perp \vec{x}_2$ at all points). Prove that the simple surface given in Exercise 1 is an orthogonal net.

8.* A simple surface $\vec{x}\colon U \to \mathbb{R}^3$ is called a *Chebyshev net* if the lengths of opposite sides of any quadrilateral formed by them are equal. That is, for any $(u_0^1, u_0^2) \in U$ and $\Delta u^1, \Delta u^2 > 0$ such that $[u_0^1, u_0^1 + \Delta u^1] \times [u_0^2, u_0^2 + \Delta u^2] \subset U$, $\vec{x}$ maps $[u_0^1, u_0^1 + \Delta u^1] \times [u_0^2, u_0^2 + \Delta u^2]$ to a quadrilateral in the image; we have a Chebyshev net if opposite sides of this quadrilateral are of equal length. Show that $\vec{x}$ is a Chebyshev net if and only if

$$\frac{\partial g_{11}}{\partial u^2} = \frac{\partial g_{22}}{\partial u^1} = 0.$$

9.* Prove that if $\vec{x}\colon U \to \mathbb{R}^3$ is a Chebyshev net (use the equations given in Exercise 8), then there is a coordinate transformation $f : \tilde{U} \to U$ such that the resulting simple surface $\tilde{\vec{x}} := \vec{x} \circ f$ has metric tensor

$$\begin{pmatrix} 1 & \cos\theta \\ \cos\theta & 1 \end{pmatrix},$$

for some function θ.

10. Consider the surface of revolution given in Exercise 1. Show that there is a coordinate transformation $f : \tilde{U} \to U$ such that the resulting simple surface $\tilde{\vec{x}} := \vec{x} \circ f$ has metric tensor

$$\begin{pmatrix} 1 & 0 \\ 0 & G(t) \end{pmatrix},$$

for some function $G(t)$.
Hint: You don't really need to think about coordinate transformations. This is just a matter of being careful with the original curve.

11. For a simple surface $\vec{x}\colon U \to \mathbb{R}^3$, show that the parameter u^1 measures arc length on the u^1-curves if and only if $g_{11} \equiv 1$.

12.* Consider the simple surface $\vec{x}\colon (0, \infty) \times (-\pi, \pi) \to \mathbb{R}^3$ defined by

$$x(r, \theta) := (r\,\cos\theta, r\,\sin\theta, r).$$

(a) Find the coefficients of the metric tensor (g_{ij}).

(b) Consider the curve

$$\vec{\alpha}(t) := \vec{x}(r(t), \theta(t)) := \vec{x}(e^{t\,(\cot\beta)/2}, t/\sqrt{2}), \quad t \in (0, \pi),$$

where β is constant. Find the length of this curve, and show that β is the angle between $\vec{\alpha}'$ and any line of the form $\theta = \theta_0$ on $\vec{x}$.

13. Let's call a simple surface $\vec{x}\colon U \to \mathbb{R}^3$ *conformal* if $d\vec{x}$ preserves angles. That is, for any point $p \in U$ and pair of vectors $\vec{u}, \vec{v} \in T_pU = \mathbb{R}^2$,

$$\measuredangle(d\vec{x}_p(\vec{u}), d\vec{x}_p(\vec{v})) = \measuredangle(\vec{u}, \vec{v}).$$

Prove that $\vec{x}$ is conformal if and only if the metric coefficients satisfy $g_{11} = g_{22}$ and $g_{12} = 0$.

14.* Let $\vec{x}\colon U \to \mathbb{R}^3$ be a simple surface, and set $\mathcal{S} := \mathrm{im}\vec{x}$. Given a differentiable function $f : \mathcal{S} \to \mathbb{R}$, we can define a differentiable vector field $\nabla f : \mathcal{S} \to \mathbb{R}^3$ by the properties $\nabla f(p) \in T_p\mathcal{S}$ and

$$\langle \nabla f(p), v \rangle = df_p(v), \quad \text{for all } v \in T_p\mathcal{S},$$

for all $p \in \mathcal{S}$. We call ∇f the *gradient* of f. Prove the following statements.

(a) The gradient is given in terms of $\vec{x}_1$ and $\vec{x}_2$ by

$$\nabla f = \frac{f_{u^1}\, g_{22} - f_{u^2}\, g_{12}}{g_{11}\, g_{22} - g_{12}^2}\vec{x}_1 + \frac{f_{u^2}\, g_{11} - f_{u^1}\, g_{12}}{g_{11}\, g_{22} - g_{12}^2}\vec{x}_2$$

(What does this tell us in case $\mathcal{S} = \mathbb{R}^2$?)

(b) If we fix a point $p \in \mathcal{S}$ and consider all vectors $\vec{v} \in T_p\mathcal{S}$ with $\|\vec{v}\| = 1$, then $df_p(\vec{v})$ is maximized precisely when $\vec{v} = \nabla f(p)/\|\nabla f(p)\|$. (So ∇f points in the direction of greatest increase.)

(c) If $\nabla f \neq 0$ at every point of some *level curve* $f^{-1}(c) := \{p \in \mathcal{S} \,|\, f(p) = c\}$, then ∇f is perpendicular to $f^{-1}(c)$ at all points of $f^{-1}(c)$.