2 Homework 2

Instructions

Complete the following exercises and upload your work to Gradescope by **11:59 pm on September 23**.

Solutions to the starred exercises do not need to be submitted, but you should know how to do them. Of the solutions you submit, some will be checked carefully while others are graded for submission only.

Be sure to **acknowledge your collaborators** and any resources you reference!

Acknowledgment: Several of the problems below are adapted from Millman & Parker's *Elements of Differential Geometry* and Shifrin's *Differential Geometry*.

1.* Verify Green's theorem for

$$\int_C \frac{x}{x^2+y^2}\,dy - \frac{y}{x^2+y^2}\,dx,$$

where C is the triangle with vertices at $(0,1)$, $(1,0)$, and $(1,1)$.

2. Let $\vec{\alpha}$ be a unit-speed plane curve, and let $\{\vec{t}(s), \vec{n}(s)\}$ denote its planar Frenet frame, while $\{\vec{T}(s), \vec{N}(s), \vec{B}(s)\}$ denotes its Frenet frame as a space curve.
   (a) Prove that $\vec{t}(s) = \vec{T}(s)$ and $\vec{n}(s) = \pm\vec{N}(s)$, whenever $\vec{N}(s)$ is defined.
   (b) Prove that $\kappa(s) = |k(s)|$, for all s in the domain of $\vec{\alpha}$.
   (c) Prove that $\vec{n}'(s) = -k(s)\,\vec{t}(s)$, for all s in the domain of $\vec{\alpha}$.

3. Show that the evolute $\vec{\varepsilon}$ of a plane curve $\vec{\alpha}$ is uniquely determined by the condition that its tangent line at each point $\vec{\varepsilon}(t)$ is the normal line to $\vec{\alpha}$ at $\vec{\alpha}(t)$.

4. Let $\vec{\alpha}\colon [0,L] \to \mathbb{R}^2$ be a simple, closed, unit-speed plane curve. Prove that the *tangent circular image* $\vec{t}\colon [0,L] \to S^1$ is onto. (Here $S^1 \subset \mathbb{R}^2$ is the unit circle centered at the origin.)
   Hint: Use the rotation index theorem.

5.* Draw closed plane curves with rotation indices $0, 1, 3, -1$, and -2, respectively.

6. Suppose C is a convex, simple, closed plane curve with maximum curvature κ_0. Prove that the distance between any pair of parallel tangent lines of C is at least $2/\kappa_0$.

7.* We can define convexity for plane curves which are piecewise C^1. Such a curve is continuous, but its derivative may have discrete discontinuities, which we call *junction points*. At each junction point, there are two tangent lines — one coming from each direction. We'll say that a piecewise C^1 curve is convex if it lies on one side of each of its tangent lines — including both tangent lines at junction points. Give an example to show that our characterization of convex curves in Activity 4 is false for piecewise C^2 curves.

The next two problems might feel a bit elementary, but we'll need them both for our proof of the isoperimetric inequality. You can (and maybe should?) do them before we discuss the isoperimetric inequality in class.

8. Let $\vec{\alpha}$ be a simple, closed plane curve whose image bounds a region $\mathcal{R} \subset \mathbb{R}^2$, and which is traversed counterclockwise. Prove that the area of $\mathcal{R}$ is given by

$$\int_{\vec{\alpha}} x\,dy = -\int_{\vec{\alpha}} y\,dx,$$

where x and y are the usual coordinates of $\mathbb{R}^2$.

9. Let a and b be positive numbers. Prove that

$$\sqrt{ab} \leq \frac{1}{2}(a+b),$$

with equality if and only if $a = b$. (This is called the **arithmetic mean-geometric mean inequality**.)

10. Suppose C is a simple, closed plane curve with $0 < \kappa \leq c$, for some constant c. Prove that the length of C is at least $2\pi/c$.

11. Consider distinct points $\vec{a}, \vec{b} \in \mathbb{R}^2$, and fix some length $\ell > \|\vec{a} - \vec{b}\|$. Consider all possible regular curve segments in $\mathbb{R}^2$ which begin at $\vec{a}$, end at $\vec{b}$, and have length ℓ. Which such curve will enclose the largest possible (signed) area when joined with the line segment from $\vec{b}$ to $\vec{a}$?
Note: Don't just recreate a proof of the isoperimetric inequality. Instead, assume the isoperimetric inequality and argue that your curve is the best one.