1 Homework 1

Instructions

Complete the following exercises and upload your work to Gradescope by **11:59 pm on September 9**.

Solutions to the starred exercises do not need to be submitted, but you should know how to do them. Of the solutions you submit, some will be checked carefully while others are graded for submission only.

Be sure to **acknowledge your collaborators** and any resources you reference!

Acknowledgment: Several of the problems below are adapted from Millman & Parker's *Elements of Differential Geometry*.

For the first exercise, you'll need to recall that a matrix M is said to be **skew-symmetric** if $M^T = -M$. The problem asks you to consider a differentiable family of $A(t)$ matrices. This means that the entries of the matrix $A(t)$ are differentiable functions of t, and we take the t-derivative of $A(t)$ entry-wise.

1. Suppose $A(t)$ is a differentiable family of orthogonal matrices. Prove that the matrix $(A(t))^{-1}A'(t)$ must be skew-symmetric.
   Hint: The product rule
   $$\frac{d}{dt}(A(t)B(t)) = A'(t)B(t) + A(t)B'(t)$$
   should be helpful.

2. Consider the following functions:
   $$\begin{aligned}\vec{\alpha}(\theta) &= (\cos\theta, 1-\cos\theta-\sin\theta, -\sin\theta)\\ \vec{\beta}(\theta) &= (2\sin^2\theta, 2\sin^2\theta\ \tan\theta, 0)\\ \vec{\gamma}(\theta) &= (\cos\theta, \cos^2\theta, \sin\theta).\end{aligned}$$
   (a) Which of these functions give regular parametrized curves?
   (b) For each curve which is regular at $\theta = \pi/4$, find an equation for the tangent line.

3. Let $g : (0,\infty) \to (0,1)$ be given by $g(r) = r^2/(r^2+1)$. Is this a reparametrization? (I should have specified the regularity required. Let's say it's enough to show that this is a C^1-reparametrization.)

4.* Let $\vec{\alpha}(\theta) = (e^\theta \cos\theta, e^\theta \sin\theta, 0)$. Prove that the angle between $\vec{\alpha}$ and $\vec{T}$ is constant. (A curve with this property is called a *logarithmic spiral*.)

5. Let $\vec{\alpha}(t)$ be a regular curve. Suppose there is a point $\vec{a} \in \mathbb{R}^3$ such that $\vec{\alpha}(t) - \vec{a}$ is orthogonal to $\vec{T}(t)$ for all t. Prove that $\vec{\alpha}(t)$ lies on a sphere.
   Hint: First determine the center of the sphere.

6. Consider the right circular helix
   $$\vec{\alpha}(t) = (r\cos t, r\sin t, ht),$$
   where $r > 0$ and $h > 0$ are some constants.
   (a) Find the arc length of this helix for $0 \le t \le 10$.
   (b) Reparametrize the helix by arc length.

7.* Reparametrize the curve $\vec{\alpha}(t) = (\cosh t, \sinh t, t)$ by arc length. (This previously had a typo in the second component.)

8.* Let $\vec{\alpha}(t)$ be a regular curve with $\|\vec{\alpha}'(t)\| = a$, where a is a positive constant. Show that if s is the arc length measured from some point, then $t = (s/a) + c$, for some constant c.

9. Show that
$$\vec{\alpha}(s) = \left(\frac{(1+s)^{3/2}}{3}, \frac{(1-s)^{3/2}}{3}, \frac{s}{\sqrt{2}} \right)$$
is a unit speed curve and compute its Frenet-Serret apparatus.
Note: For fun, consider using `frenet-serret.nb` *to animate the apparatus!*

10.* Let $\vec{\alpha}(s)$ be a planar C^k curve. (That is, we can write $\vec{\alpha}(s) = (x(s), y(s), 0)$, for some differentiable functions $x(s)$ and $y(s)$.) Prove that if $\kappa \neq 0$, then $\tau = 0$.
Hint: Show that $\vec{B} = \pm(0,0,1)$.

11.* Let $\vec{\alpha}(s) = (x(s), y(s), 0)$ be a unit speed curve. Prove that
$$\kappa = \|x'\,y'' - x''\,y'\|.$$

12. Prove that $\kappa\,\tau = -\langle \vec{T}', \vec{B}' \rangle$.

For the last two exercises, we'll need to define tangent, normal, and binormal **spherical images**. We'll come back to these later in the course.

If $\vec{\alpha}(s)$ is a unit speed curve, then we can define a new curve
$$\vec{T} : (a,b) \to \mathbb{R}^3$$
whose input is $s \in (a,b)$ and output is $\vec{T}(s)$, computed from α. Because $\|\vec{T}(s)\| = 1$ by our definition of $\vec{T}$, the image of this curve lies on the sphere of radius 1 about the origin. We call this curve the **tangent spherical image** of $\vec{\alpha}$. Note that *the tangent spherical image might not be regular*! That is, $\vec{T}'(s)$ might sometimes be $\vec{0}$.

We can similarly define a curve $s \mapsto \vec{N}(s)$ which we call the **normal spherical image** of $\vec{\alpha}$ and a curve $s \mapsto \vec{B}(s)$ which we call the **binormal spherical image** of $\vec{\alpha}$.

13. Find the tangent, normal, and binormal spherical images of the helix in problem 6.

14.* (Challenge) Let $\bar{s}$ be arc length along the tangent spherical image of $\vec{\alpha}$, so that
$$\bar{s} = \int_0^s \|\vec{T}'(u)\|\,du.$$

(a) Prove that $d\bar{s}/ds = \kappa$.

(b) Find a necessary and sufficient condition for the tangent spherical image of $\vec{\alpha}$ to be a regular curve.

(c) Let s^* be the arc length along the normal spherical image of α. Prove that $ds^*/ds = \sqrt{\kappa^2 + \tau^2}$.

(d) Let $\tilde{s}$ be the arc length along the binormal spherical image of α. Prove that $d\tilde{s}/ds = |\tau|$.