Math 4441 **December 5, 2022**

LAST TIME

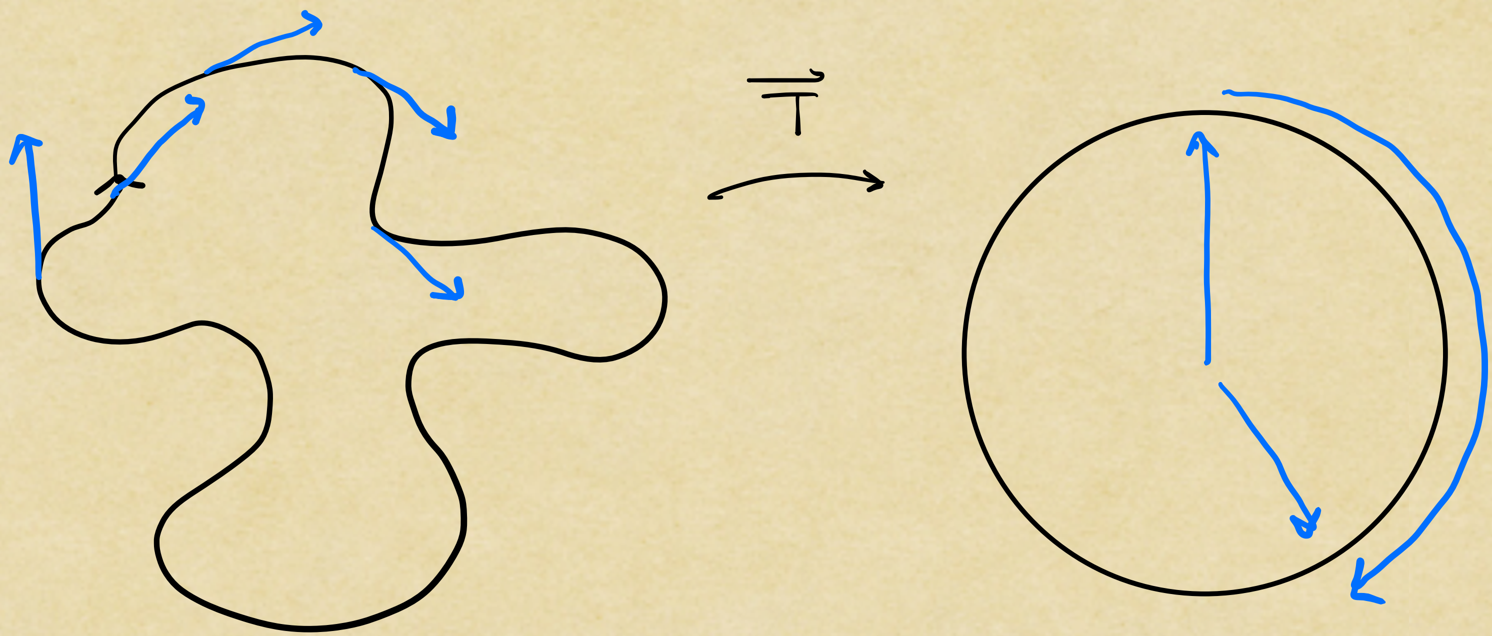
We proved a remarkable theorem: Gaussian curvature is intrinsic!

TODAY

Generalizing the rotation index to surfaces gives a beautiful link between geometry and topology.

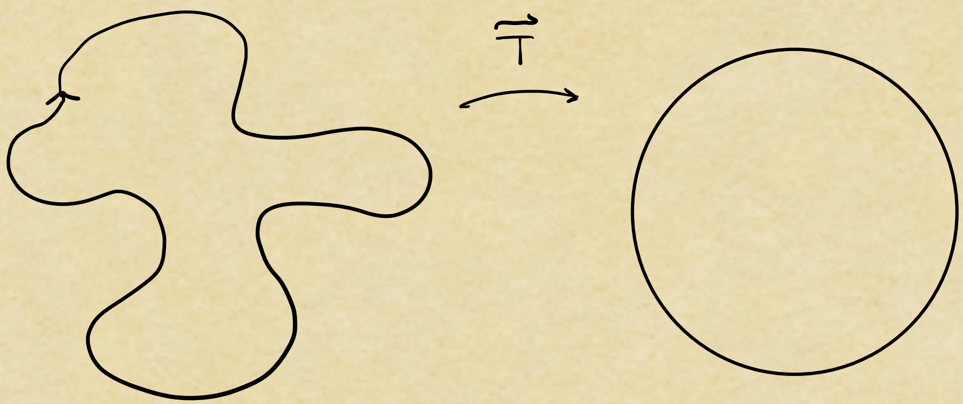
Recall

For closed planar curves, the rotation index counts the number of CCW rotations:

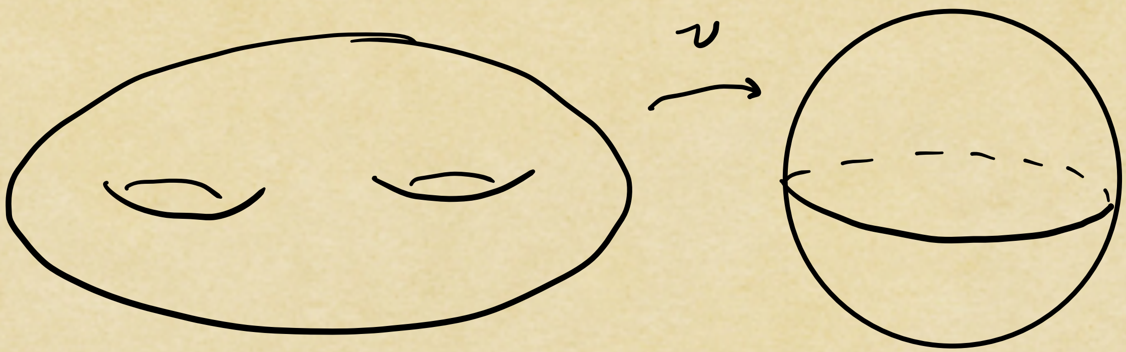


After some work, we found the rotation index to be $\frac{1}{2\pi}\int_{\vec{\alpha}} k\, ds$.

For surfaces, there's no obvious "rotation" to measure, but we can still integrate curvature.



$$\frac{1}{2\pi}\int_{\vec{\alpha}} k\,ds$$

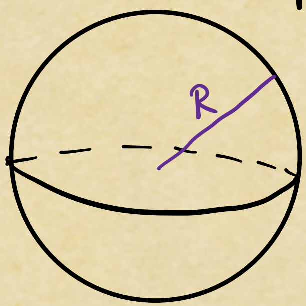


? ?

? $\iint_{\vec{x}} K\,dA$?

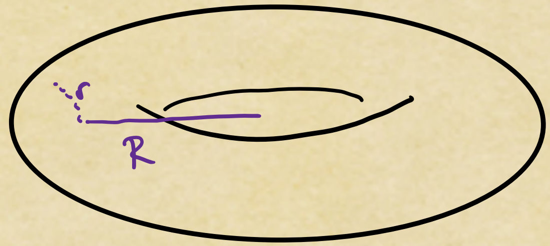
? ?

Some examples



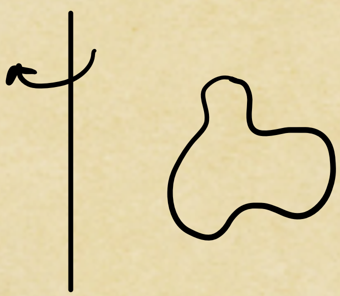
$$\iint_{\vec{x}} K\,dA = 4\pi$$

(indep. of R)

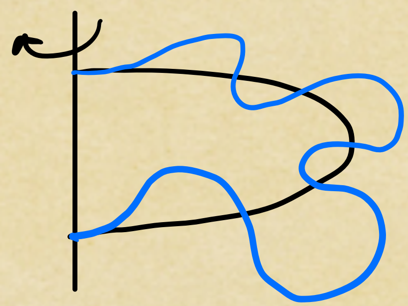


$$\iint_{\vec{x}} K\,dA = 0$$

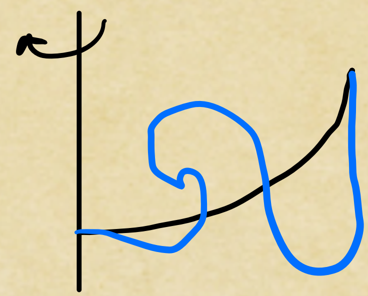
(indep. of R, r)



$$\iint_{\vec{x}} K\,dA = 0$$



$$\iint_{\vec{x}} K\,dA = 4\pi$$

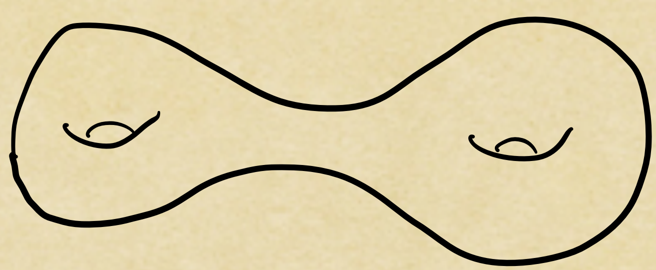


$$\iint_{\vec{x}} K\,dA = 2\pi$$

Observation: $\iint_{\vec{x}} K\,dA$ seems to depend only on the topology of $\mathrm{im}(\vec{x})$.

(Some sort of quantization is taking place.)

Any guesses as to the value of

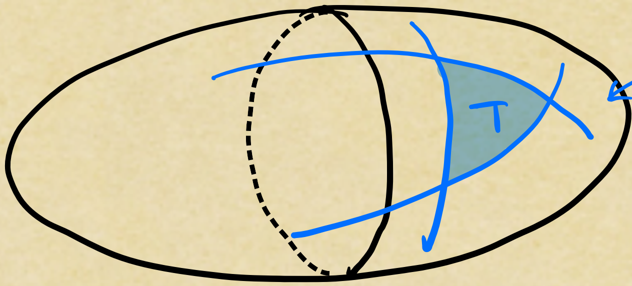


$\iint_{\vec{x}} K\,dA$ for this surface?

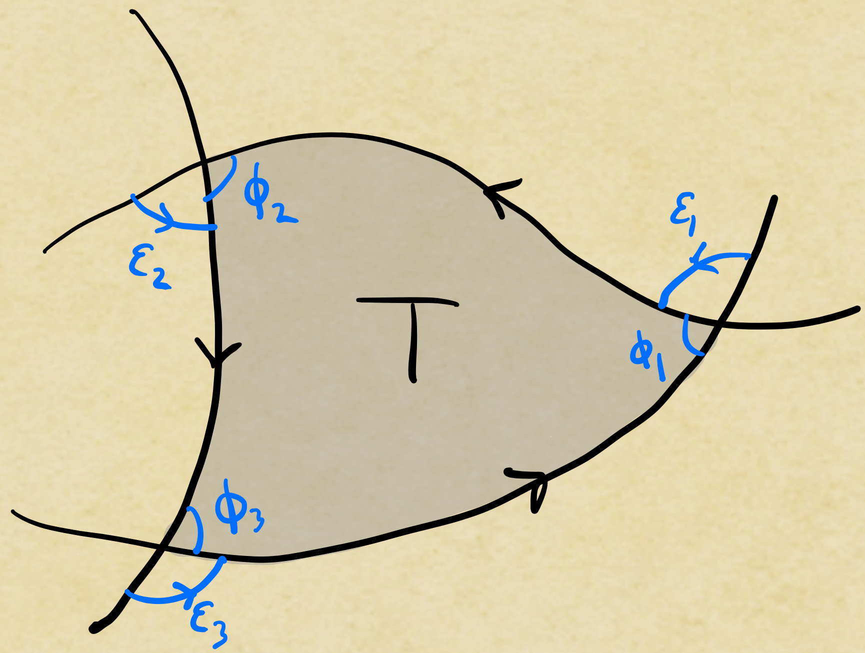
Probably a multiple of 2π?

More precisely:

Let T be a "triangle" in a surface.



There are three **exterior angles**: $\varepsilon_1, \varepsilon_2, \varepsilon_3$.



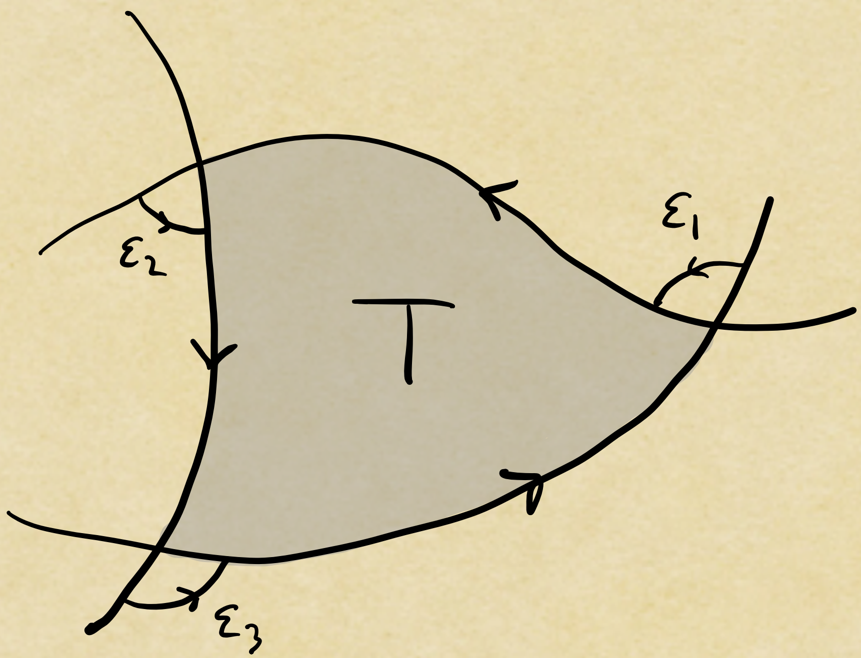
Note that

$$\varepsilon_i = \pi - \phi_i .$$

More precisely:

Fact. For any such triangle T,

$$\boxed{\iint_T K\,dA + \oint_{\partial T} k_g\,ds + \sum_{i=1}^{3} \varepsilon_i = 2\pi}$$



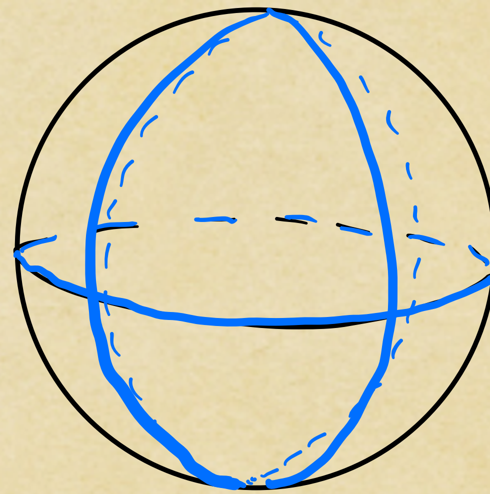
We can prove this using Stokes' theorem, but don't have time.

:(

Now suppose we have a compact surface S.

A triangulation $\mathcal{T}$ of S is a decomposition of S into triangles:

i.e., $S = \bigcup_{T \in \mathcal{T}} T$

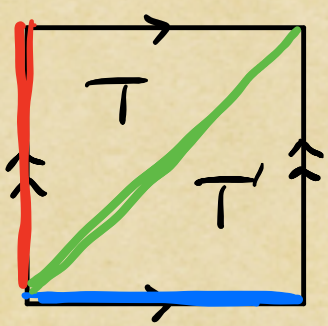


$|\mathcal{T}| = 8$

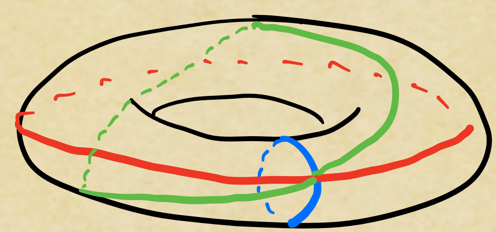
and for any $T, T' \in \mathcal{T}$,

$T \cap T' = \partial T \cap \partial T' =$ a single edge

Ex



$\rightsquigarrow$



$|\mathcal{T}| = 2$

If we have a triangulation $\mathcal{T}$ of S, we can apply our formula

$$\iint_T K\,dA + \oint_{\partial T} k_g\,ds + \sum_{i=1}^{3} \varepsilon_i = 2\pi$$

to each $T \in \mathcal{T}$:

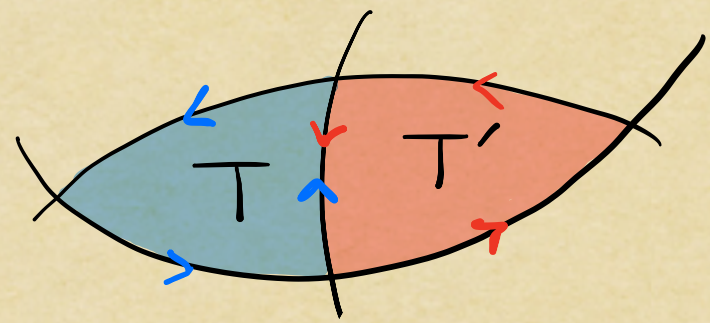
$$\sum_{T\in\mathcal{T}} \left(\iint_T K\,dA + \oint_{\partial T} k_g\,ds + \sum_{i=1}^{3} \varepsilon_i \right) = 2\pi|\mathcal{T}|$$

So $$\sum_{T\in\mathcal{T}} \iint_T K\,dA + \sum_{T\in\mathcal{T}} \oint_{\partial T} k_g\,ds + \sum_{T\in\mathcal{T}} \sum_{i=1}^{3} \varepsilon_i = 2\pi|\mathcal{T}|$$

(Let's assume $\underline{\partial S = \emptyset}$ for now.)

Let's consider the terms of

$$\underbrace{\sum_{T\in\mathcal{T}} \iint_T K\,dA}_{\iint_S K\,dA} + \underbrace{\sum_{T\in\mathcal{T}} \oint_{\partial T} k_g\,ds} + \sum_{T\in\mathcal{T}} \sum_{i=1}^{3} \varepsilon_i = 2\pi|\mathcal{T}|$$

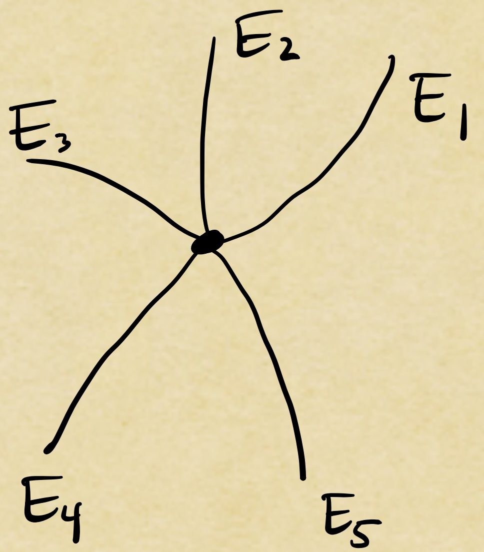


$$\sum_{T\in\mathcal{T}} \oint_{\partial T} k_g\,ds = 0$$

Then $$\iint_S K\,dA + \sum_{T\in\mathcal{T}} \sum_{i=1}^{3} \varepsilon_i = 2\pi|\mathcal{T}| \qquad (\star)$$

The term $\sum_{T\in\mathcal{T}}\sum_{i=1}^{3}\varepsilon_i$ requires some care.

Let's investigate one vertex at a time.



Here, the interior angles sum to 2π. But for each adjacent edge,

interior $\measuredangle$ + exterior $\measuredangle = \pi$.

So the sum of the exterior angles is

$$\sum_{\substack{\text{adj.}\\ \text{edges}}} \text{exterior } \measuredangle = \sum_{\substack{\text{adj.}\\ \text{edges}}} (\pi - \text{interior } \measuredangle) = \sum_{\substack{\text{adj.}\\ \text{edges}}} \pi - \sum_{\substack{\text{adj.}\\ \text{edges}}} \text{interior } \measuredangle$$

$$= \pi \cdot \begin{pmatrix} \#\text{ adj.} \\ \text{edges} \end{pmatrix} - 2\pi$$

Knowing that the exterior angle sum for each vertex is $\left(\begin{smallmatrix}\#\text{ adj.}\\ \text{edges}\end{smallmatrix}\right)\cdot\pi - 2\pi$, the sum over *all* vertices is

$$\sum_{T\in\mathcal{T}}\sum_{i=1}^{3}\varepsilon_i = 2\pi(\#\text{edges}) - 2\pi(\#\text{vertices})$$

$$= 2\pi(E-V)$$

each edge is adjacent to 2 vertices

Plugging into our big formula (☆):

$$\iint_S K\,dA + 2\pi(E-V) = 2\pi|\mathcal{T}|$$

This gives us

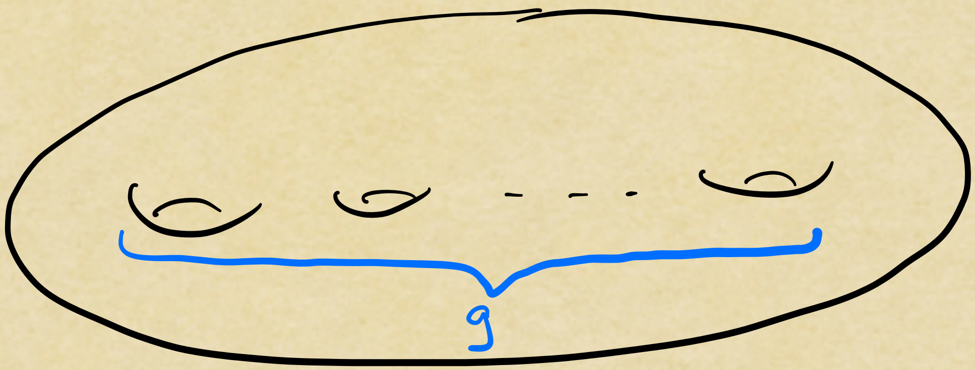
Theorem (Gauss–Bonnet) For any triangulation $\mathcal{T}$ of a compact surface S with $\partial S = \emptyset$,

$$\iint_S K \, dA = 2\pi(F - E + V) =: 2\pi \chi(S),$$

where $F = \#$ faces of $\mathcal{T}$,
$E = \#$ edges of $\mathcal{T}$,
$V = \#$ vertices of $\mathcal{T}$.

Euler characteristic

Example Let Σ_g be a surface of genus g:



Then $\chi(\Sigma_g) = 2 - 2g$, so

$$\iint_{\Sigma_g} K \, dA = 2\pi(2-2g) = 4\pi(1-g),$$

regardless of the geometry!

It turns out that two closed (nice) surfaces $S, S' \subset \mathbb{R}^3$ are diffeomorphic if and only if $\underline{\chi(S) = \chi(S')}$.

Since K is $\underline{\text{intrinsic}}$, this means that we can determine the $\underline{\text{topological type}}$ of a surface from the $\underline{\text{first fundamental form}}$ alone!

What if there's boundary?

Theorem (Gauss-Bonnet) For any triangulation $\mathcal{T}$ of a compact surface S

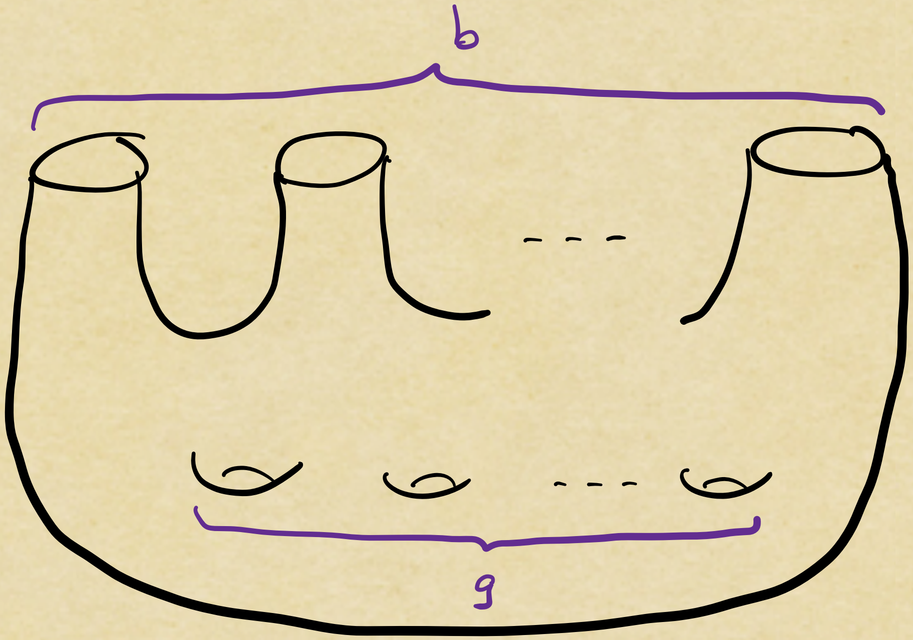
$$\iint_S K\, dA + \oint_{\partial S} k_g\, ds = 2\pi(F-E+V),$$

where $F = \#$ faces of $\mathcal{T}$,

$E = \#$ edges of $\mathcal{T}$,

$V = \#$ vertices of $\mathcal{T}$.

Ex $\Sigma_{g,b} =$



$$\chi(\Sigma_{g,b}) = 2 - 2g - b$$

$$\therefore \quad \iint_{\Sigma_{g,b}} K\, dA + \oint_{\partial \Sigma_{g,b}} k_g\, ds = 4 - 4\pi g - 2\pi b$$