Math 4441 August 31, 2022

LAST TIME

We defined the <u>Frenet-Serret apparatus</u>

$$\{\kappa(s), \tau(s), \vec{T}(s), \vec{N}(s), \vec{B}(s)\}$$

for a unit-speed curve $\vec{\alpha}$, provided <u>$\kappa(s) \neq 0$</u>.

TODAY

How does the Frenet frame change w.r.t. s, expressed <u>in the frame itself</u>?

Can we <u>recover the curve</u> from the Frenet-Serret apparatus?

Throughout, $\vec{\alpha}$ is a unit-speed curve in $\mathbb{R}^3$.

When we're moving along $\vec{\alpha}$, our preferred basis for $\mathbb{R}^3$ at $\vec{\alpha}(s_0)$ is $\underline{\{\vec{T}(s_0), \vec{N}(s_0), \vec{B}(s_0)\}}$, <u>not</u> $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$.

Our first goal is to express $\vec{T}'$, $\vec{N}'$, & $\vec{B}'$ in this preferred basis.

Theorem (The Frenet-Serret equations)

For a unit-speed curve $\vec{\alpha}(s)$ with $\kappa(s) \neq 0$, we have

$$\begin{aligned}
\vec{T}'(s) &= \kappa(s)\vec{N}(s) \\
\vec{N}'(s) &= -\kappa(s)\vec{T}(s) + \tau(s)\vec{B}(s) \\
\vec{B}'(s) &= -\tau(s)\vec{N}(s)
\end{aligned}$$

That is,

$$\begin{pmatrix} \vec{T}' \\ \vec{N}' \\ \vec{B}' \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix}}_{\text{skew-symm. matrix}} \begin{pmatrix} \vec{T} \\ \vec{N} \\ \vec{B} \end{pmatrix}.$$

Our proof will need the following:

Lemma. Let V be an n-dimensional inner product space, and let $E = \{\vec{e}_1, \dots, \vec{e}_n\}$ be any set of n orthonormal vectors in V. Then

① E is a basis;

② any vector $\vec{v} \in V$ can be expressed as

$$\vec{v} = \sum_{i=1}^{n} \langle \vec{v}, \vec{e}_i \rangle \vec{e}_i$$

(Pf.) Exercise. $\diamond$

(Proof of Frenet-Serret equations)

Recall that $\vec{N} := \dfrac{\vec{T}'}{\kappa}$, so

$$\vec{T}' = \kappa \cdot \vec{N}$$

is free. To express $\vec{N}'$ in terms of $\{\vec{T}, \vec{N}, \vec{B}\}$, we'll use the lemma:

$$\vec{N}' = \langle \vec{T}, \vec{N}' \rangle \vec{T} + \langle \vec{N}, \vec{N}' \rangle \vec{N} + \langle \vec{B}, \vec{N}' \rangle \vec{B}$$

So we need to compute $\langle \vec{T}, \vec{N}' \rangle$, $\langle \vec{N}, \vec{N}' \rangle$, and $\langle \vec{B}, \vec{N}' \rangle$. For each, we'll use the <u>product rule</u>.

Let's start with $\langle \vec{N}, \vec{N}' \rangle$:

$$\langle \vec{N}, \vec{N} \rangle \equiv 1 \Rightarrow \langle \vec{N}', \vec{N} \rangle + \langle \vec{N}, \vec{N}' \rangle = 0$$

$$\therefore 2 \langle \vec{N}, \vec{N}' \rangle = 0$$

What about $\langle \vec{T}, \vec{N}' \rangle$?

$$\langle \vec{T}, \vec{N} \rangle \equiv 0 \Rightarrow \langle \vec{T}', \vec{N} \rangle + \langle \vec{T}, \vec{N}' \rangle = 0$$

$$\begin{aligned} \therefore \langle \vec{T}, \vec{N}' \rangle &= - \langle \vec{T}', \vec{N} \rangle \\ &= - \langle \kappa \vec{N}, \vec{N} \rangle \\ &= -\kappa \langle \vec{N}, \vec{N} \rangle \end{aligned}$$

So $\langle \vec{T}, \vec{N}' \rangle = -\kappa$.

Finally, $\langle \vec{B}, \vec{N}' \rangle$.

$$\langle \vec{B}, \vec{N} \rangle \equiv 0 \implies \langle \vec{B}', \vec{N} \rangle + \langle \vec{B}, \vec{N}' \rangle = 0$$

$$\langle \vec{B}, \vec{N}' \rangle = -\langle \vec{B}', \vec{N} \rangle$$

$$= -(-\tau)$$

So $\langle \vec{B}, \vec{N}' \rangle = \tau$

Altogether,

$$\vec{N}' = \langle \vec{T}, \vec{N}' \rangle \vec{T} + \langle \vec{N}, \vec{N}' \rangle \vec{N} + \langle \vec{B}, \vec{N}' \rangle \vec{B}.$$

$$= -\kappa \cdot \vec{T} + \tau \vec{B} \quad ,$$

as desired.

Exercise: Verify that $\vec{B}' = -\tau \vec{N}$.

(You'll want to keep using the product rule) $\diamond$

These formulas allow us to treat the Frenet frame as our preferred basis, even when doing calculus.

Note: For any orthonormal frame $F = \{\vec{f}_1, \vec{f}_2, \vec{f}_3\}$ along $\vec{\alpha}(s)$, we have

$$F'(s) = A \cdot F(s),$$

for some skew-symmetric matrix A.

(e.g., $A = \underline{\ 0\ }$ if $F = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ is std.)

What makes $\{\vec{T}, \vec{N}, \vec{B}\}$ special is the particular contents of A.

__Corollary.__ For any unit speed curve $\vec{\alpha}(s)$ with $\kappa(s) \neq 0$, the following are equivalent:

① $\vec{\alpha}$ is contained in a single plane of $\mathbb{R}^3$;

② $\vec{B}$ is __constant__;

③ τ __$\equiv 0$__.

(Proof.) Let's start with ② $\Leftrightarrow$ ③. From the theorem, __$\vec{B}' = -\tau\vec{N}$__, so __$\vec{B}$ is constant iff $\tau \equiv 0$__

Next, ① $\Rightarrow$ ②. We can write an equation for the plane P containing $\vec{\alpha}$:

$$(x,y,z) \in P \iff \langle (x,y,z) - \vec{x}_0, \vec{n} \rangle = 0,$$

for some fixed point $\vec{x}_0$ and constant vector $\vec{n}$.

So $\langle \vec{\alpha}(s) - \vec{x}_0, \vec{n} \rangle = 0$, if $\vec{\alpha}$ is planar.

Differentiating, $\langle \vec{\alpha}'(s), \vec{n} \rangle = 0$

& $\langle \vec{\alpha}''(s), \vec{n} \rangle = 0$

Since $\vec{T}$ is parallel to $\underline{\vec{\alpha}'}$ and $\vec{N}$ is parallel to $\underline{\vec{\alpha}''}$,

so $\langle \vec{T}, \vec{n} \rangle = 0$ & $\langle \vec{N}, \vec{n} \rangle = 0 \Rightarrow \vec{B} \parallel \vec{n}$.

So $\vec{n}$ is perpendicular to both $\vec{T}$ & $\vec{N}$, just like $\vec{B}$. Then

$$\vec{B} = \pm \frac{\vec{n}}{|\vec{n}|} \quad \Rightarrow \quad \vec{B}' = \vec{0}.$$

Finally, ② ⇒ ①. If we show that

$$\langle \vec{\alpha}(s) - \vec{\alpha}(s_0), \vec{B} \rangle \equiv 0,$$

for some fixed s_0, we'll have that $\vec{\alpha}$ lives in the plane through $\vec{\alpha}(s_0)$ with normal vector $\vec{B}$.

The equation is certainly true when $s = s_0$.

To show that it's always true, differentiate.

$\vec{0}$, b/c $\vec{B}$ is const

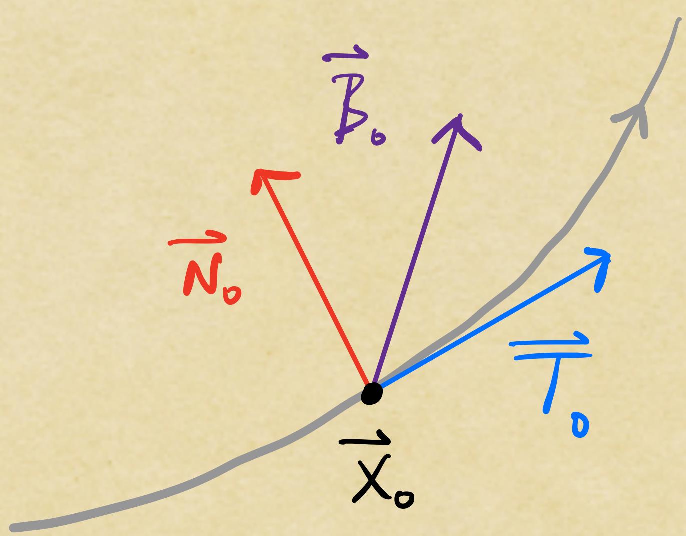
$$\frac{d}{ds}\left(\langle \vec{\alpha}(s) - \vec{\alpha}(s_0), \vec{B}\rangle\right) = \langle \vec{\alpha}'(s), \vec{B}\rangle + \langle \vec{\alpha}(s) - \vec{\alpha}(s_0), \cancel{\vec{B}'}\rangle$$

$$= \langle \vec{\alpha}'(s), \vec{B}\rangle = \langle \vec{T}, \vec{B}\rangle$$

$$= 0$$

So the LHS is constant, meaning that the equation is always true. $\diamond$

Finally, let's see what can be reverse-engineered from the Frenet-Serret apparatus.



Say we're at a point $\vec{X}_0$ and have an ONB $\{\vec{T}_0, \vec{N}_0, \vec{B}_0\}$.

If we pick scalar functions $\kappa(s) > 0$, $\tau(s)$,

Can we find $\vec{\alpha}(s)$, unit-speed, with the correct curvature, torsion, and Frenet frame?

The fundamental theorem of space curves

$\boxed{\text{\underline{Theorem}. Any regular curve with } \kappa > 0 \text{ is completely determined by its curvature and torsion functions, up to a choice of initial position and frame. i.e., an isometry}}$

(Pf.) Appeal to Picard's theorem for existence and uniqueness of solutions to ODEs. ◇

Upshot: If we know $\{\vec{T}(s), \vec{N}(s), \vec{B}(s)\}$ for all s, then we know $\vec{\alpha}(s)$, up to choosing a starting point. This is because the Frenet-Serret equations allow us to recover κ & τ

So the Frenet frame is perfectly adapted to $\vec{\alpha}$!