Math 4441 August 24, 2022

LAST TIME

- Regular, parametrized curves and their reparametrizations
- Our goal: geometric invariants

TODAY

- Tangent vectors as a geometric invariant.
- Arc length as a geometric invariant.

Recall from last time:

A geometric invariant of a regular, parametrized curve $\vec{\alpha}$ is any mathematical quantity which doesn't change when we replace $\vec{\alpha}$ with a reparametrization $\vec{\alpha} \circ g$.

So we'll use the function $\vec{\alpha}$ to study $\mathrm{im}(\vec{\alpha})$ — but if $\vec{\beta} = \vec{\alpha} \circ g$, then we want our calculations to work for $\vec{\beta}$, too.

Tangent vectors as geometric invariants

The first quantity we computed from a curve $\vec{\alpha}(t)$ was the tangent vector. Let's convince ourselves that this is NOT a geometric invt.

Say that $\vec{\beta}(u)$ is a reparametrization of $\vec{\alpha}(t)$. Then $\vec{\beta}(u) = (\vec{\alpha} \circ g)(u)$, so

$$\vec{\beta}'(u) = \vec{\alpha}'(g(u)) \cdot g'(u).$$

The tangent vectors only match if $g'(u) = 1$.

The basic problem is that traversing the same curve in $\mathbb{R}^3$ at a different speed will give a different tangent vector.

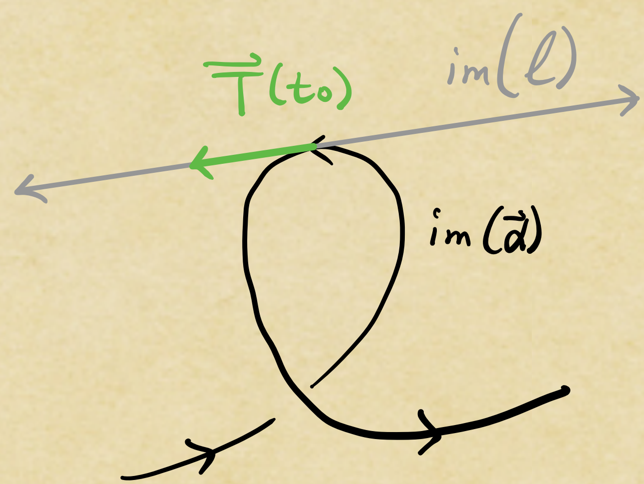
To fix this, consider:

> The (unit) tangent vector field of a regular curve $\vec{\alpha}(t)$ is the v.v.f. defined by
>
> $$\vec{T}(t) := \frac{1}{|\vec{\alpha}'(t)|}\,\vec{\alpha}'(t)$$

While we're at it, we can define

The tangent line of (or to) a regular curve $\vec{\alpha}: (a,b) \to \mathbb{R}^3$ at $t_0 \in (a,b)$ is the line $\ell: \mathbb{R} \to \mathbb{R}^3$ defined by

$$\ell(\lambda) := \underline{\vec{\alpha}(t_0) + \lambda \vec{T}(t_0)}.$$



Proposition. The unit tangent vector of a regular curve is a geometric invariant, up to a sign*, and thus the tangent line is a geometric invariant.

* i.e., reparametrizing might multiply by ± 1

(Proof.) Given a regular, parametrized curve $\vec{\alpha}:(a,b)\to\mathbb{R}^3$ and a reparametrization $g:(c,d)\to(a,b)$, consider $\vec{\beta}:=\vec{\alpha}\circ g$. We need to check that

$$\vec{T}_{\vec{\alpha}}(g(u)) = \pm\vec{T}_{\vec{\beta}}(u)$$

We've already computed $\vec{\beta}'(u)$ today:

$$\vec{\beta}'(u) = (\vec{\alpha} \circ g)'(u) = \vec{\alpha}'(g(u)) \cdot g'(u)$$

From this we see that

$$\vec{T}_{\vec{\beta}}(u) = \frac{1}{|\vec{\beta}'(u)|} \cdot \vec{\beta}'(u) = \frac{g'(u)}{|\vec{\alpha}'(g(u)) \cdot g'(u)|} \vec{\alpha}'(g(u))$$

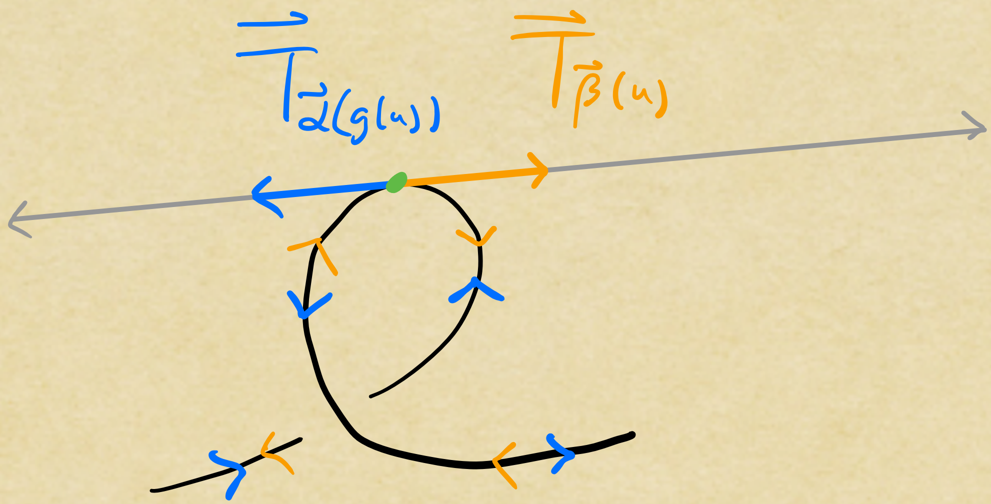
$$= \frac{g'(u)}{|g'(u)|} \cdot \frac{\vec{\alpha}'(g(u))}{|\vec{\alpha}'(g(u))|}$$

Need $g'(u) \neq 0$. Already proved that today.

$$= \frac{g'(u)}{|g'(u)|} \cdot \vec{T}_{\vec{\alpha}}(g(u)) = \pm 1 \cdot \vec{T}_{\vec{\alpha}}(g(u)).$$

Since $\vec{T}_{\vec{\alpha}}(g(u)) = \pm \vec{T}_{\vec{\beta}}(u)$, we must have

$$\operatorname{im}\left(\ell_{\vec{\alpha}(g(u))}\right) = \operatorname{im}\left(\ell_{\vec{\beta}(u)}\right).$$



$\diamond$

Example 1 For any reparametrization

$$g : (c,d) \longrightarrow (0, 2\pi),$$

the curve

Prob. should have used u.

$$\vec{\alpha}(t) = (\cos(g(t)), \sin(g(t)), 0)$$

will have image equal to the unit circle in the xy-plane centered at the origin.

Let's compute $\vec{T}(t)$ at $(-1, 0, 0)$.

$$\vec{\alpha}\,'(t) = (-g'(t)\sin(g(t)), g'(t)\cos(g(t)), 0)$$

Example 1, cont'd

$$\vec{\alpha}'(t) = \left(-g'(t)\sin(g(t)),\ g'(t)\cos(g(t)),\ 0\right)$$

$$|\vec{\alpha}'(t)|^2 = (-g'(t))^2 \sin^2(g(t)) + (g'(t))^2 \cos^2(g(t))$$

$$= (g'(t))^2 \longrightarrow |\vec{\alpha}'(t)| = |g'(t)|$$

$$\therefore\ \vec{T}(t) = \frac{\vec{\alpha}'(t)}{|\vec{\alpha}'(t)|}$$

$$= \frac{g'(t)}{|g'(t)|}\left(-\sin(g(t)),\ \cos(g(t)),\ 0\right)$$

$$= \pm 1 \cdot \left(-\sin(g(t)),\ \cos(g(t)),\ 0\right)$$

Arc length

A good thing to define in a geometry course is length.

Given a regular curve $\vec{\alpha}:(\tilde{a},\tilde{b}) \to \mathbb{R}^3$ and a closed subinterval $[a,b] \subset (\tilde{a},\tilde{b})$, the **arc length** of the "curve segment" $\vec{\alpha}:[a,b] \to \mathbb{R}^3$ is given by $\underline{\int_a^b |\vec{\alpha}'(t)|\,dt}$.

<u>Proposition</u>. Arc length is a geometric invariant of curve segments.

(Proof). We'll prove this by taking

- regular curve seg. $\vec{\alpha} : [a,b] \longrightarrow \mathbb{R}^3$;
- reparametrization $g : [c,d] \to [a,b]$

and checking that

$$\int_c^d |(\vec{\alpha} \circ g)'(u)|\, du = \int_a^b |\vec{\alpha}'(t)|\, dt.$$

The arc length of $\vec{\beta}^{\,= \vec{\alpha}\circ g}$ from c to d is

$$\int_c^d |\vec{\beta}'(u)|\, du$$

Let's make a substitution : $\quad t = g(u)$
$\quad dt = g'(u)\, du$

$u = c \to t = g(c)$
$u = d \to t = g(d)$

$$\int_{g(c)}^{g(d)} |\vec{\beta}'(u)| \frac{dt}{g'(u)}$$

This contains some nonsense!

Two cases: $\left.\begin{array}{l} g'(u) > 0 \\ g'(u) < 0 \end{array}\right\}$ $g'(u)$ is never zero

Case 1 $g'(u) > 0 \Rightarrow \begin{array}{l} g(c) = a \\ g(d) = b \end{array}$ & $\dfrac{|g'(u)|}{g'(u)} = 1$

So $$\int_{g(c)}^{g(d)} |\vec{\beta}'(u)| \frac{dt}{g'(u)} = \int_a^b |\vec{\alpha}'(t)| \cdot |g'(u)| \cdot \frac{dt}{g'(u)} = \int_a^b |\vec{\alpha}'(t)|\, dt$$

Case 2 $g'(u) < 0 \Rightarrow \begin{array}{l} g(c) = b \\ g(d) = a \end{array}$ & $\dfrac{|g'(u)|}{g'(u)} = -1$

So $$\int_{g(c)}^{g(d)} |\vec{\beta}'(u)| \frac{dt}{g'(u)} = \int_b^a |\vec{\alpha}'(t)| \cdot \frac{|g'(u)|}{g'(u)}\, dt = \int_a^b |\vec{\alpha}'(t)|\, dt$$

$\diamond$

Read Remark 1.13 of Willis notes!

Arc length parametrizations

A quantity is a geometric invariant if and only if it doesn't change under reparametrization.

But it's annoying to check this every time we define a new invariant.

Solution: Only define invariants for unit-speed curves (i.e., those with $|\vec{\alpha}'(t)| \equiv 1$).

So the steps for computing a geometric invariant of an arbitrary regular curve $\vec{\alpha}$ are (theoretically)

1. Reparametrize so that we have unit speed.

2. Compute the invariant for unit-speed curves.

In practice, 1 can be quite messy. We call a unit-speed parametrization of $\vec{\alpha}$ an <u>arc length parametrization.</u>

Algorithm for parametrizing by arc length

① Choose an <u>initial time</u> $t_0 \in (a,b)$.

② Define a <u>length function</u>

t was already taken

$$s = h(t) = \int_{t_0}^{t} |\vec{\alpha}'(u)|\, du$$

③ Compute the <u>inverse</u> of h:

$$t = g(s) = h^{-1}(s)$$

④ Define the new parametrization: $\vec{\beta}(s) := (\vec{\alpha} \circ g)(s)$.

Remarks

- From now on, we'll usually assume our curves are unit-speed.
- Read the statement and proof of Thm 1.14 of Willis' notes. (Really, do this!)
- Here's a helpful fact: If $\vec{\beta}(s)$ is a unit-speed curve, then

$$\vec{T}(s) = \frac{\vec{\beta}'(s)}{|\vec{\beta}'(s)|} = \vec{\beta}'(s).$$