Math 4441 August 22, 2022

GOALS

1. Define regular, parametrized curves.
2. See some examples.
3. Define reparametrization.
4. Identify the goal: geometric invariants.

Nice functions

We call a function $f: \mathbb{R}^m \to \mathbb{R}^n$ a C^k function, for some $k \geq 1$, if f has

continuous derivatives of all orders up to k

e.g., $f: \mathbb{R}^2 \to \mathbb{R}$ is C^2 ← Changed from 3 for less writing if $f, f_x, f_y, f_{xx}, f_{xy}, f_{yx}, \text{ \& } f_{yy}$ are all cts

Notation: $f \in C^k(\mathbb{R}^m, \mathbb{R}^n)$

Curves as functions

A parametrized curve in $\mathbb{R}^3$ is a C^3 function

$$\vec{\alpha} : (a, b) \longrightarrow \mathbb{R}^3.$$

We say that $\vec{\alpha}$ is regular if $\underline{\vec{\alpha}'(t) \neq 0}$, for all $t \in (a, b)$.

Notation: $\vec{\alpha}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = (x(t), y(t), z(t))$

$$\Rightarrow \vec{\alpha}'(t) = \frac{d\vec{\alpha}}{dt} = \begin{pmatrix} x'(t) \\ y'(t) \\ z'(t) \end{pmatrix} = (x'(t), y'(t), z'(t))$$

Remarks / Vocabulary

1. We'll often refer to the independent variable t as time.

2. The derivative $\vec{\alpha}'(t)$ is called the velocity vector of $\vec{\alpha}$.

   Warning: Need to choose a value of t to get a vector!

3. The speed of $\vec{\alpha}$ is given by $|\vec{\alpha}'(t)|$.

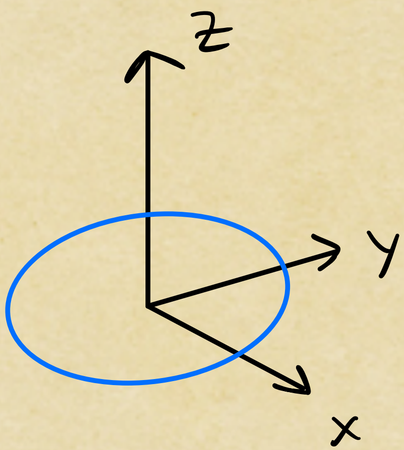
Example 1 Describe the curve

$$\vec{\alpha} : (0, 2\pi) \longrightarrow \mathbb{R}^3$$

defined by $\vec{\alpha}(t) := (2\cos t, 2\sin t, 0)$.

Plot $\vec{\alpha}'(t)$ at $t = \pi/2, \pi, 3\pi/2$.

circle of radius 2 centered @ origin, in xy-plane



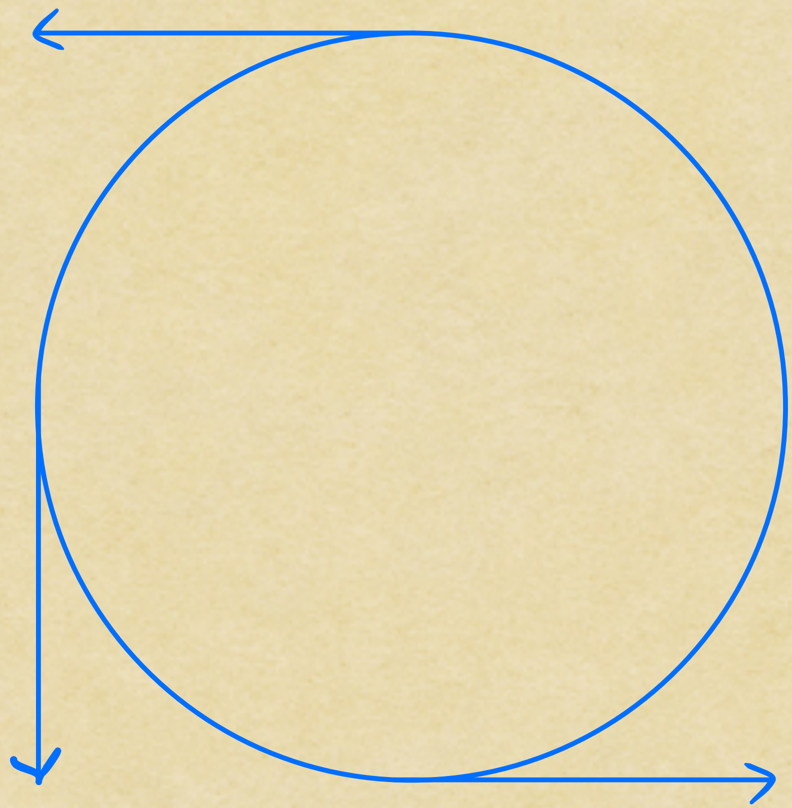
$$\vec{\alpha}'(t) = (-2\sin t, 2\cos t, 0)$$

$$\vec{\alpha}'(\pi/2) = (-2, 0, 0)$$

$$\vec{\alpha}'(\pi) = (0, -2, 0)$$

$$\vec{\alpha}'(3\pi/2) = (2, 0, 0)$$

Example 1, cont'd



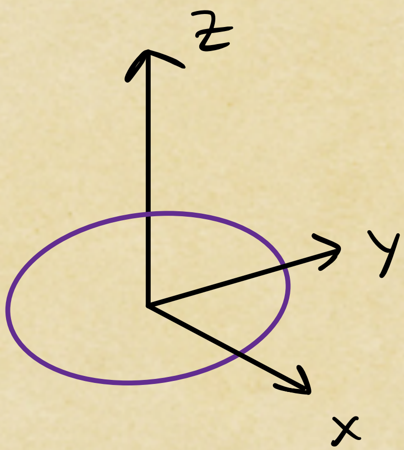
Example 2 Describe the curve

$$\vec{\beta} : (0, 6\pi) \longrightarrow \mathbb{R}^3$$

defined by $\vec{\beta}(u) := \left(2\cos\left(\frac{u}{3}\right), 2\sin\left(\frac{u}{3}\right), 0\right)$.

Plot $\vec{\beta}'(u)$ at $u = 3\pi/2,\ 3\pi,\ 9\pi/2$.

circle of radius 2 centered @ origin, in xy-plane



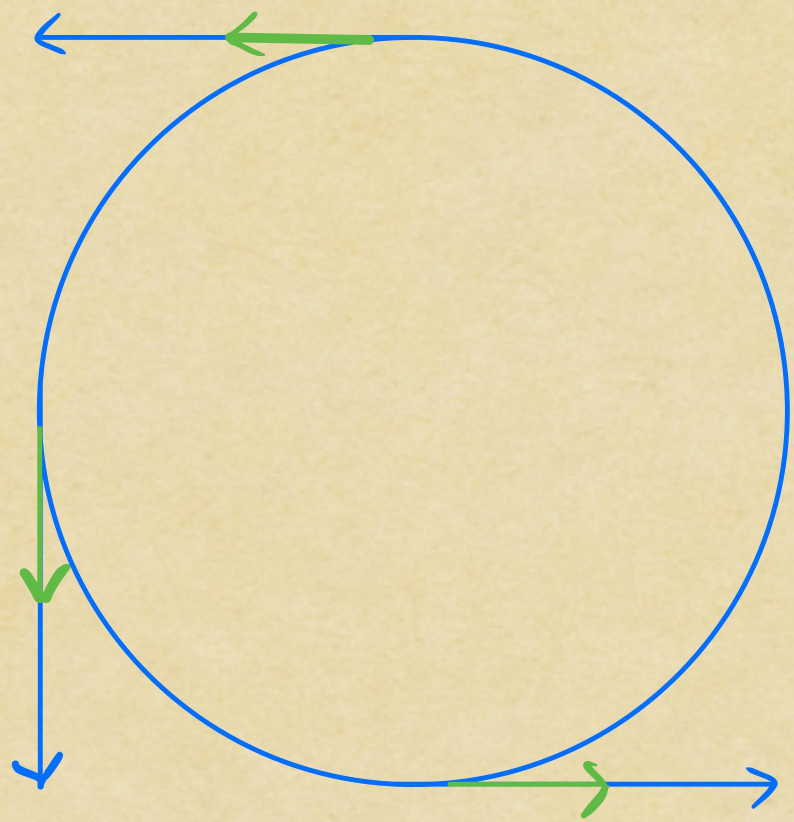
$$\vec{\beta}'(u) = \left(-\frac{2}{3}\sin\left(\frac{u}{3}\right), \frac{2}{3}\cos\left(\frac{u}{3}\right), 0\right)$$

$$\vec{\beta}'\left(\frac{3\pi}{2}\right) = \left(-\frac{2}{3}, 0, 0\right)$$

$$\vec{\beta}'(3\pi) = \left(0, -\frac{2}{3}, 0\right)$$

$$\vec{\beta}'\left(\frac{9\pi}{2}\right) = \left(\frac{2}{3}, 0, 0\right)$$

Example 2, cont'd



Didn't make it very far ⸚

Curves as functions

The preceding examples illustrate an important point: while treating curves as functions is computationally helpful, we want to study the images of these functions.

We need a way to treat $\vec{\alpha}$ and $\vec{\beta}$ as the same curve.

Reparametrizations

A (1-dimensional) C^k reparametrization is a bijective function

$$g : (c,d) \longrightarrow (a,b)$$

such that g & g^{-1} are both C^k functions.

Note: For curves we need $k \geq 3$

We'll usually take $k = \infty$

Example 3 Consider $\vec{\alpha} : (0, 2\pi) \longrightarrow \mathbb{R}^3$
$t \longmapsto (2\cos t, 2\sin t, 0)$
$\vec{\beta} : (0, 6\pi) \longrightarrow \mathbb{R}^3$
$u \longmapsto \left(2\cos\left(\frac{u}{3}\right), 2\sin\left(\frac{u}{3}\right), 0\right)$

from before. Find a reparametrization $g : (c,d) \to (a,b)$ such that $\vec{\beta} = \vec{\alpha} \circ g$.

$$g : (0, 6\pi) \longrightarrow (0, 2\pi)$$
$$u \longmapsto u/3$$

$$(\vec{\alpha} \circ g)(u) = \vec{\alpha}(g(u)) = \underbrace{\vec{\alpha}(u/3)}_{\vec{\alpha}\text{, but replace } t \text{ with } u/3}$$

By design, reparametrizing does not change the image of a curve.

<u>Proposition.</u> Let $\vec{\alpha} : (a,b) \to \mathbb{R}^3$ be a regular curve, $g : (c,d) \to (a,b)$ a reparametrization. Then

$$\vec{\beta} := \underline{\vec{\alpha} \circ g} : (c,d) \to \mathbb{R}^3$$

is a regular curve, and $\underline{\mathrm{im}(\vec{\alpha}) = \mathrm{im}(\vec{\beta})}$.

In the above circumstance, we say that $\vec{\beta}$ is a <u>reparametrization of $\vec{\alpha}$</u>

(Proof.) We'll have $\mathrm{im}(\vec{\beta}) = \mathrm{im}(\vec{\alpha})$ because g is <u>surjective</u>.

We need to check that $\vec{\beta}$ is <u>regular</u>.

Let's use the variables $t \in (a,b)$ & $u \in (c,d)$.

Then

$$\frac{d\vec{\beta}}{du} = \frac{d}{du}\left(\vec{\alpha}\circ g\right) = \frac{d\vec{\alpha}}{dt}(g(u)) \cdot \frac{dg}{du},$$

and we need <u>$\frac{d\vec{\beta}}{du} \neq 0$</u>. By assumption, <u>$\frac{d\vec{\alpha}}{dt} \neq 0$</u>,

so it's enough to check that <u>$\frac{dg}{du} \neq 0$</u>.

For this, we can use the chain rule again.

For any $u \in (c,d)$, we have

$$(g^{-1} \circ g)(u) = u \quad \Rightarrow \quad \frac{d}{du}(g^{-1} \circ g) = 1.$$

On the other hand,

$$1 = \frac{d}{du}(g^{-1} \circ g) = \frac{dg^{-1}}{dt} \cdot \frac{dg}{du},$$

so $\dfrac{dg}{du} \neq 0$, as desired.

reparametrizations ♥ chain rule

$\diamond$

Here's a useful result that we won't prove.

Theorem. If $\vec{\alpha}$ and $\vec{\beta}$ are injective, regular, parametrized curves with $\operatorname{im}(\vec{\alpha}) = \operatorname{im}(\vec{\beta})$, then there is a reparametrization g such that

$$\vec{\beta} = \vec{\alpha} \circ g.$$

Idea: Set $g = \vec{\alpha}^{-1} \circ \vec{\beta}$.

Geometric invariants

Our first goal in this course is to study images of curves. The preceding results tell us that this means studying parametrized curves, up to reparametrization.

A geometric invariant of a regular, parametrized curve $\vec{\alpha}$ is any mathematical quantity which doesn't change when we replace $\vec{\alpha}$ with a reparametrization $\vec{\alpha} \circ g$.