9 Clairaut's relation

Goals

By the end of this activity, we should be able to do the following.

1. State **Clairaut's relation**, which gives a qualitative characterization of geodesics on a surface of revolution.
2. Prove Clairaut's relation using the geodesic differential equations derived in class.
3. Use Clairaut's relation to identify geodesics on surfaces of revolution — including finding all geodesics on a right circular cylinder.

Acknowledgment: Today's activity draws heavily from Pressley's *Elementary Differential Geometry*.

The goal of today's activity will be to characterize the geodesics on any surface of revolution. Thankfully, we've already computed a lot of information about these surfaces. Throughout, we'll work with the simple surface $\vec{x}\colon (a,b)\times(-\pi,\pi)$ defined by

$$\vec{x}(t,\theta) := (r(t)\cos\theta, r(t)\sin\theta, z(t)),$$

where $r\colon (a,b)\to(0,\infty)$ and $z\colon (a,b)\to\mathbb{R}$ are smooth functions. Moreover, we'll assume that $(r(t), z(t))$ parametrizes a unit-speed curve in the rz-plane, so that $\dot{r}^2+\dot{z}^2=1$.

We've previously computed the matrix of metric coefficients:

$$(g_{ij}) = \begin{pmatrix} \dot{r}^2+\dot{z}^2 & 0 \\ 0 & r^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}.$$

The second equation uses our assumption that $\dot{r}^2+\dot{z}^2=1$.

Since our plan is to talk about geodesics, we're going to need some second-derivative information (the first fundamental form gives us first-derivative information). Specifically, we need the Christoffel symbols. But recall that the Christoffel symbols are in fact determined by the matrix of metric coefficients.

Proposition 9.1. *The Christoffel symbols satisfy*

$$\Gamma_{ij}^k = \frac{1}{2}\sum_{\ell=1}^{2} g^{\ell k}\left[\frac{\partial g_{i\ell}}{\partial u^j} - \frac{\partial g_{ij}}{\partial u^\ell} + \frac{\partial g_{j\ell}}{\partial u^i}\right],$$

for $1\le i,j,k\le 2$.

Exercise 9.1. Use Proposition 9.1 to show that the Christoffel symbols of a surface of revolution satisfy

$$\Gamma_{22}^1 = -r\,\dot{r} \quad \text{and} \quad \Gamma_{12}^2 = \Gamma_{21}^2 = \frac{\dot{r}}{r},$$

with all others being 0.
Hint: A useful first step is to show that $\frac{\partial g_{22}}{\partial u^1}$ *is the only nonzero derivative of the form* $\frac{\partial g_{ij}}{\partial u^\ell}$*. Then write the* $k=1$ *and* $k=2$ *versions of Proposition 9.1 separately.*

Okay, we're ready to start thinking about curves on our surface of revolution, so let's set some notation for that. We'll think about a unit-speed surface curve

$$\vec{\alpha}(s) = \vec{x}(t(s), \theta(s)),$$

$s\in(c,d)$, so that the "downstairs curve" is $\vec{\alpha}_U(s) = (t(s),\theta(s))$, $s\in(c,d)$. In terms of the component notation used in our book and in class, this means that

$$\alpha_U^1(s) = t(s) \quad \text{and} \quad \alpha_U^2(s) = \theta(s).$$

Remember that we're using the parameter s because the *upstairs* curve $\vec{\alpha}$ is unit-speed, but we cannot assume that the downstairs curve $\vec{\alpha}_U$ is unit-speed.

In class we characterized geodesics with a differential equation involving the downstairs curve and the Christoffel symbols.

Proposition 9.2. *Let $\vec{\alpha} = \vec{x} \circ \vec{\alpha}_U$ be a unit-speed surface curve. Then $\vec{\alpha}$ is a geodesic if and only if*

$$(\alpha_U^k)'' + \sum_{i,j=1}^{2} (\alpha_U^i)'(\alpha_U^j)'\,\Gamma_{ij}^k = 0,$$

for $k = 1, 2$.

Let's rewrite these for our particular setting.

Exercise 9.2. Let $\vec{\alpha}(s) = \vec{x}(t(s), \theta(s))$ be a unit-speed surface curve as above. Show that $\vec{\alpha}$ is a geodesic if and only if

$$t'' - (\theta')^2\, r(t)\, \dot{r}(t) = 0 \quad \text{and} \quad \theta'' + 2t'\,\theta'\,\frac{\dot{r}(t)}{r(t)} = 0, \tag{9.1}$$

where the prime denotes derivative with respect to s and the dot denotes derivative with respect to t.
Hint: Write down the ODE from Proposition 9.2 for $k = 1$ and $k = 2$ separately, and then sub in the Christoffel symbols from Exercise 9.1.

We'll try to extract some useful information from these differential equations shortly, but first let's verify a couple of families of geodesics.

Exercise 9.3. Use Equation 9.1 to show that

(a) every meridian $\theta = \theta_0$ is a geodesic;

(b) a latitude $t = t_0$ is a geodesic if and only if $\dot{r}(t_0) = 0$.

Hint: The meridians and latitudes are the u^1- and u^2-curves, so you should be able to come up with $\vec{\alpha}_U(s) = (t(s), \theta(s))$ without too much pain.

Exercise 9.4. Based on Exercise 9.3, sketch a surface of revolution for which exactly three latitudes are geodesics. Sketch these three geodesic latitudes, plus at least two more geodesics.

It's great that we can describe a couple of families of geodesics, but we still don't have *all* geodesics. Consider a point p on your sketch which lies at the intersection of two geodesics. If we pick some direction which is neither horizontal nor vertical (so that it's not tangent to either of the geodesics you've drawn), then we know from our existence and uniqueness theorem that a geodesic exists in this direction, but do we know anything about it? Finding it would require solving the differential equations in Equation 9.1, which seems hard. But we can at least learn some things about the geodesic.

Our first step is to prove a property that's enjoyed by every unit-speed surface curve on $\vec{x}$ — geodesic or not. In order to do this, we want to think about the angle that our curve makes with the latitudes of our surface.

Exercise 9.5. Let $\vec{\alpha}(s) = \vec{x}(t(s), \theta(s))$ be a unit-speed surface curve as above. Explain why, for each $s \in (c, d)$, there must be some angle $\psi(s)$ such that

$$\vec{\alpha}'(s) = \sin\psi(s)\,\vec{x}_t + \cos\psi(s)\left(\frac{1}{r(t(s))}\,\vec{x}_\theta\right).$$

Hint: We like expressing vectors in orthonormal bases.

On the other hand, we can use the chain rule to differentiate $\vec{\alpha}(s) = \vec{x}(t(s), \theta(s))$:

$$\vec{\alpha}'(s) = t'(s)\,\vec{x}_t + \theta'(s)\,\vec{x}_\theta.$$

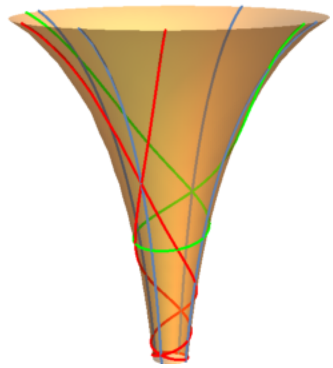

Figure 9.1: Some geodesics on a pseudosphere. All meridians (blue) are geodesics and, for this surface, none of the latitudes are geodesics. For all geodesics, the quantity $r\,\cos\psi$ is constant.

Comparing this with the result of Exercise 9.5 — specifically, comparing the coefficient of $\vec{x}_\theta$ in these two expressions — we see that $\theta'(s) = \cos\psi(s)/r(t(s))$. Multiplying both sides by r^2 yields

$$(r(t(s)))^2\,\theta'(s) = r(t(s))\,\cos\psi(s). \tag{9.2}$$

So far we haven't made any assumptions about $\vec{\alpha}$ being a geodesic, so Equation 9.2 holds for any unit-speed surface curve on $\vec{x}$. In case $\vec{\alpha}$ *is* a geodesic, this leads to a very nice property.

Exercise 9.6. Let $\vec{\alpha} = \vec{x}(t(s), \theta(s))$ be as above, and let $\psi(s)$ be as defined in Exercise 9.5. Prove that if $\vec{\alpha}$ is a geodesic, then the quantity $r\,\cos\psi$ is constant.
Hint: Instead of differentiating the quantity $r(t(s))\cos\psi(s)$, use Equation 9.2 to replace this with something more familiar. Eventually you'll need to use Equation 9.1.

We've made a good observation about geodesics: the angle ψ that they make with the latitudes they cross will only change when the distance r from the axis of rotation is changing. But our goal was to find new geodesics, so it would be more helpful to have a converse. The best we can do is a partial converse, as stated in the following theorem.

Theorem 9.3: Clairaut's relation

Let $\vec{\alpha}$ be a unit-speed surface curve on a surface of revolution $\vec{x}$, and let $r : \operatorname{im}(\vec{x}) \to (0, \infty)$ measure the distance from the axis of revolution. Let $\psi(s)$ give the angle made by $\vec{\alpha}'(s)$ with the latitudes of $\vec{x}$. If $\vec{\alpha}$ is a geodesic, then $r\,\cos\psi$ is constant. Conversely, if $r\,\cos\psi$ is constant and no portion of $\vec{\alpha}$ lies on a latitude, then $\vec{\alpha}$ is a geodesic.

Remark. There's a physical interpretation of Clairaut's relation which corresponds to the fact that a body traveling along a geodesic on a surface of revolution will have constant angular momentum about the axis of revolution.

To prove the theorem, we need to show that if the quantity $r\,\cos\psi$ is constant and no portion of $\vec{\alpha}$ lies on a latitude — that is, no interval along with $\vec{\alpha}$ is a latitude — then $\vec{\alpha}$ is a geodesic. We'll do this by showing that t and θ satisfy the differential equations given in Equation 9.1. Your work in Exercise 9.6 should establish the second of those two ODEs, so it remains to check the first. This is slightly trickier, so we'll break it up.

Exercise 9.7. Show that, for any surface curve $\vec{\alpha}(s) = \vec{x}(t(s), \theta(s))$,

$$(t')^2 = 1 - (r\,\theta')^2 = 1 - \left(\frac{r\,\cos\psi}{r}\right)^2, \tag{9.3}$$

where ψ is the angle described above.
Hint: Think about how we came up with Equation 9.2.

The right hand side of Equation 9.3 looks a little fishy, but remember our assumption in the converse case of Clairaut's relation: we're assuming that $r\cos\psi$ is equal to some constant K. So we have $(t')^2 = 1-(K/r)^2$.

Exercise 9.8. Differentiate both sides of $(t')^2 = 1-(K/r)^2$ with respect to s to show that

$$t'\left(t'' - \frac{K^2}{r^3}\dot{r}\right) = 0.$$

Hint: You'll need to remember the chain rule when differentiating r with respect to s.

Now we chose the constant K so that $K = r\cos\psi = r^2\,\theta'$, so what you've really shown in Exercise 9.8 is that

$$t'\left(t'' - \frac{r^4(\theta')^2}{r^3}\dot{r}\right) = 0 \quad\Rightarrow\quad t'\left(t'' - (\theta')^2\, r\,\dot{r}\right) = 0.$$

Since we can't have $t' = 0$ over any interval, the expression in parentheses must be zero. But this is precisely the first ODE from Equation 9.1! So $\vec{\alpha}$ is a geodesic, and we've established Clairaut's relation.

Exercise 9.9. Use Clairaut's relation to identify all geodesics on the right circular cylinder $x^2 + y^2 = 1$. You should find three infinite families of geodesics, and support your claim with some complete sentences.

Exercise 9.10. (*Optional*) Suppose a surface of revolution is asymptotic to its axis of revolution. See Figure 9.1. Prove that the meridians are the only geodesics on this surface which are asymptotic to the axis of revolution.